Annex of Additional Notes

B: Orthogonality of the Spherical Harmonics

We show only that $Y^m_l$ and $Y^n_k$ are orthogonal if the degrees are different, that is, $l \neq k$; we will discuss the problem of $l = k$ and $m \neq n$ at the end. The proof goes as follows: first we show the important proposition that eigenfunctions of any self-adjoint linear operator corresponding to different eigenvalues are orthogonal and the eigenvalues real. Then we show the surface Laplacian (denoted by $\nabla^2_1$) is self-adjoint; this takes us into generalizations of vector calculus identifies for operators on curved manifolds, and Gauss’ Theorem for the same.

Orthogonality of Eigenfunctions of a Self-Adjoint Operator.

First we will introduce a notation used everywhere in advanced mathematics: the inner product. The inner product of two functions will be defined for our purposes by:

$$\langle f, g \rangle = \int d^2\hat{s} \, f(\hat{s}) \, g(\hat{s})^*$$

where the integral will be over the whole unit sphere, $S(1)$ unless otherwise indicated. It is obviously a complex number and satisfies:

$$\langle g, f \rangle = \langle f, g \rangle^*.$$  \hspace{1cm} (B.2)

The inner product is the generalization to function spaces of the ordinary dot product between two vectors, and can be thought of that way for most purposes. For us it is a notational shorthand. You might think using parentheses would be confusing since they already have a perfectly good use; strangely, confusion is rare, perhaps because the comma isn’t very popular, except in certain tensor notations.

Consider a self adjoint linear operator $L$ acting on functions over $S(1)$. In the new notation, self adjointness looks like this: for every $f$ and $g$ on the sphere

$$\langle Lf, g \rangle = \langle f, Lg \rangle.$$  \hspace{1cm} (B.3)

This is the property equivalent to the symmetry of real matrices on finite dimensional spaces, but of course you can’t see that it is true as easily as you can with a three by three matrix. We first show eigenvalues of $L$ are real (we already know that when $L = \nabla^2_1$ the eigenvalues are $-l(l+1)$ for nonnegative integer $l$, which is a set of real numbers). Suppose

$$Lu_1 = \lambda_1 u_1, \quad u_1 \neq 0$$

and this means $u_1$ does not vanish everywhere on $S(1)$. In (B.4) take the inner product with $u_1$

$$\langle u_1, Lu_1 \rangle = \lambda_1^* \langle u_1, u_1 \rangle.$$  \hspace{1cm} (B.5)

Notice from (B.2) [or (B.1) if you prefer] that $\langle u_1, u_1 \rangle$ is always a real number. Next take the complex conjugate of (B.5) and apply (B.2) on the left:
If \( L \) is self-adjoint (B.3) shows that the left sides of (B.5) and (B.6) are equal, so upon subtracting these equations we see:

\[
0 = (\lambda_1^* - \lambda_1)(u_1, u_1) .
\]

(B.7)

Since \( u_1 \) cannot be identically zero, the inner product is not zero; we conclude \( \lambda_1^* = \lambda_1 \), so that \( \lambda_1 \) must be real.

Our next task is to show that for \( L \) with two unequal eigenvalues, \( \lambda_1 \) and \( \lambda_2 \), the corresponding eigenfunctions \( u_1 \) and \( u_2 \) are orthogonal. In inner product language, we wish to show \( (u_1, u_2) = 0 \). Now we have two eigenvalue equations

\[
Lu_1 = \lambda_1 u_1
\]

(B.8)

\[
Lu_2 = \lambda_2 u_2
\]

(B.9)

and \( \lambda_1 \neq \lambda_2 \). Take the inner product of \( u_2 \) with (B.8):

\[
(u_2, Lu_1) = \lambda_1^*(u_2, u_1) .
\]

(B.10)

Then use \( u_1 \) with (B.9):

\[
(u_1, Lu_2) = \lambda_2^*(u_1, u_2) .
\]

(B.11)

Take the complex conjugate of (B.11) and apply (B.3):

\[
(Lu_2, u_1) = \lambda_2(u_2, u_1) .
\]

(B.12)

Now subtract (B.7) from (B.5):

\[
(u_2, Lu_1) - (Lu_2, u_1) = (\lambda_1^* - \lambda_2)(u_2, u_1) .
\]

(B.13)

The left side is zero from (B.2). Since the eigenvalues are real (B.13) states

\[
(\lambda_1 - \lambda_2)(u_2, u_1) = 0 .
\]

(B.14)

The eigenvalues are unequal by hypothesis, and neither \( u_1 \) nor \( u_2 \) vanishes identically, so the only way we can have (B.14) is if the inner product of the two eigenfunctions is zero. That means they are orthogonal.

So now we know two properties of every self-adjoint linear operator: (a) the eigenvalues are all real; (b) the eigenfunctions associated with different eigenvalues are orthogonal. Our next task is to show that \( \nabla_1^2 \) is self-adjoint.

**Self-Adjointness of the Surface Laplacian**

We examine the inner product

\[
(f, \nabla_1^2 g) = \int d^2s \ f \ \nabla_1^2 g^* .
\]

(B.15)

Notice we will prove self-adjointness for any pair of functions \( f \) and \( g \), even though we really only need the result for eigenfunctions. To make progress we use the only vector calculus identity you need to remember:
\[ \nabla \cdot (a \mathbf{A}) = \nabla a \cdot \mathbf{A} + a \nabla \cdot \mathbf{A} \cdot \]  

But there is a twist here. We need this to be true not for the ordinary div and grad that act on the Cartesian components of a vector in ordinary three-space – we want this to be true for the surface gradient \( \nabla_1 \) and its related operators \( \nabla_1 \cdot \) and \( \nabla_1^2 = \nabla_1 \cdot \nabla_1 \). (The subscript 1 is to remind us these operators work on the unit sphere.) One laborious but straight-forward proof is by expanding the operators in spherical polar coordinates (as given in any book on vector calculus) and simply verifying the truth of (B.16); I will spare you the details. What this proof misses is the point that surface operators can be defined on any sufficiently smooth simply-connected surface (ellipsoid, tomato, etc), and (B.16) will be true there too. The more general proof is in *Foundations of Geomagnetism*, Chapter 7. You may be confident that all the common vector identities are true for their surface relatives, except that surface curl is defined in a peculiar way, so it has somewhat different properties; see *Foundations* if you are interested.

To continue: in (B.16) applied to \( \nabla_1 \) etc, set \( a = f \) and \( \mathbf{A} = \nabla_1 g^* \); then rearranging:

\[ f \nabla_1^2 g^* = \nabla_1 \cdot (f \nabla_1 g^*) - \nabla_1 f \cdot \nabla_1 g^* \cdot \]  

Of course what we do next is integrate (B.17) over \( S(1) \) to get (B.15). The next result we need is that the first term on the right of (B.17) integrated over the unit sphere is always zero:

\[ \int d^2 \hat{s} \nabla_1 \cdot (f \nabla_1 g^*) = 0 \cdot \]  

This is by no means obvious, but let us accept it for now, proving it later. With this result in hand we have from (B.17) and (B.1):

\[ (f, \nabla_1^2 g) = - \int d^2 \hat{s} \nabla_1 f \cdot \nabla_1 g^* \cdot \]  

Interchanging the functions \( f \) and \( g \) in (B.19) gives:

\[ \begin{align*}
(g, \nabla_1^2 f) &= - \int d^2 \hat{s} \nabla_1 g \cdot \nabla_1 f^* \\
&= - \int d^2 \hat{s} (\nabla_1 g^* \cdot \nabla_1 f)^* \cdot \nabla_1^2 f, \quad \text{B.21}
\end{align*} \]

Now take the complex conjugate of (B.21) and apply (B.2) to the inner product on the left side:

\[ (\nabla_1^2 f, g) = - \int d^2 \hat{s} \nabla_1 g^* \cdot \nabla_1 f \cdot \]  

Finally, comparing (B.22) and (B.19) we see the right sides are equal and so this gives us the self-adjointness of \( \nabla_1^2 \):

\[ (\nabla_1^2 f, g) = (f, \nabla_1^2 g) \cdot \]  

Next we prove (B.18) which introduces an unusual form of Gauss’ Theorem.
Application of Gauss' Theorem on a Curved Surface

To obtain (B.18) requires us to apply Gauss' Theorem. It will be obvious that ordinary Gauss' Theorem over a region $B$ within $\mathbb{R}^3$ with smooth boundary $\partial B$

$$\int_B d^3s \nabla \cdot \mathbf{A}(s) = \int_{\partial B} d^2s \hat{n} \cdot \mathbf{A}(s) \quad \text{(B.24)}$$

is also true if reduced from three dimensions to points in a plane:

$$\int_C d^2s \nabla \cdot \mathbf{A}(s) = \int_{\partial C} ds \hat{n} \cdot \mathbf{A}(s) \quad \text{(B.25)}$$

where the integral on the right in (B.25) is a line integral around the perimeter of the region $C$; note that $\hat{n}$ is the outward pointing normal to $\partial C$. Less obvious is that (B.25) remains true when the region $C$ is curved surface like $S(1)$ and the operators are the surface operators $\nabla_1$, etc. Consider a simple patch $P$ on $S(1)$ and $\partial P$ be its perimeter; we assert

$$\int_{P \subseteq S(1)} d^2s \nabla_1 \cdot \mathbf{A}(\hat{s}) = \int_{\partial P} ds \hat{n} \cdot \mathbf{A}(\hat{s}) . \quad \text{(B.26)}$$

To prove this we apply the normal Gauss' Theorem (B.24) to a region $Q$ consisting of spherical shell with the shape of $P$ on $S(1)$ and extending above that surface by a small height $h$; the walls of $Q$ are locally in the radial direction. Within $Q$ make $\mathbf{A}$ a tangent vector field (one with no radial component) and constant in the radial direction. Then in the limit as $h$ tends to zero, we recover (B.26). A more general proof, valid for other kinds of curved surfaces, can be found in Chapter 7 of Foundations.

To obtain (B.18) imagine letting the patch $P$ grow larger and larger on the unit sphere until almost the whole surface is covered by $P$ and the boundary $\partial P$ has become a very short closed curve. It is clear that the integral on the right (which is a number) is less in magnitude than $L |A|_{\text{max}}$ where $L$ is the length of the boundary curve. By shrinking the boundary to a point and thereby covering the whole of $S(1)$ we show that

$$\int_{S(1)} d^2s \nabla_1 \cdot \mathbf{A}(\hat{s}) = 0 \quad \text{(B.27)}$$

for any smooth tangent vector field $\mathbf{A}$ confined to the sphere. So in (B.27) if we set $\mathbf{A} = f \nabla_1 g^*$ we have (B.18) as required.
Orthogonality over Order

We have shown that if two spherical harmonics have different degrees (that’s \( l \)) they will be orthogonal, because they come from different eigenvalues. But what about two functions of the same degree and different orders? This situation is analogous to the case of a body with two principal moments of inertia equal: say, \( A = B < C \). Then the eigenvectors corresponding to the smaller eigenvalues are not fixed in space: any vector in the equatorial plane will do as an eigenvector. This is degeneracy, but in the spherical harmonics for degree \( l \), instead of just a two-fold degeneracy, there is a \( 2l + 1 \) dimensional space of functions, any element of which is an eigenfunction. Then the proof of orthogonality fails because in (B.14) one can have zero without the inner product having to vanish. So in fact the conventional orthogonality of \( Y_{l}^{m} \) over \( m \) is an artificial choice that depends on the particular way in which the eigenfunctions are calculated and is not a general property of the eigensolutions.

With the construction of the \( Y_{l}^{m} \) in the form

\[
Y_{l}^{m} = N_{lm} P_{l}^{m}(\cos \theta) e^{im\phi} \tag{B.28}
\]

it is obvious that the functions are orthogonal over \( m \); of course we still have to show that (B.28) really is an eigenfunction of \( \nabla_{l}^{2} \), but that is a matter of lengthy algebra which I prefer to avoid. The other thing we haven’t proved is the existence of the \( 2l + 1 \)-fold degeneracy. Again, see Foundations Chapter 3 for one approach.

Exercises

B1. Present explicitly a set of \( 2l + 1 \) linearly independent eigenfunctions of \( \nabla_{l}^{2} \) with the eigenvalue \(-l(l+1)\) that are not orthogonal to each other. Within a fixed coordinate system, explain how to obtain a set of mutually orthogonal eigenfunctions of degree \( l \) that are not in the form of (B.28).

B2. This annex of additional notes provides you with all the tools to be able to prove (135) in the regular class notes. Do so.

B3. Prove property 8 in the table for large \( l \). Start with property 2 and assume the wavelength is so much smaller than 1 that the spherical curvature can be neglected locally.

The harmonic \( \text{Re} \ Y_{l}^{l} \) seems to be at variance with this property: around the equator there are \( 2l \) zeros and this looks like a wavelength of \( 2\pi/l \), not \( 2\pi/(l + \frac{1}{2}) \); at or near the poles, the distance between the zero lines of \( \text{Re} \ Y_{l}^{l} \) is much less than \( 2\pi/(l + \frac{1}{2}) \). Explain why these two objections are unfounded.