SIO 230 Geophysical Inverse Theory 2009

Supplementary Notes

8. A New Magnetic Problem

In GIT the first few chapters revolve around a marine magnetic problem based on a small set of artificial "data", supposedly collected at the North Pole. This example is too simple these days for two reasons: first, the number of observations (thirty) is trivially small; second, the idealization of the problem covers up so many of the real difficulties, particularly the questions surrounding the proper form of the forward problem. I want to work with a more realistic illustration, based on actual data, to give you a better idea of how to handle real-life situations that you may come across. However, I still want to stick to a one-dimensional geometry, to keep the graphics simple. It is another marine magnetic problem, like the one in GIT, except that the observations were taken near the sea floor in 1995 on a deep-tow vehicle (the "fish") built by the Marine Physical Lab at SIO. The profile runs across the Juan de Fuca rise (35°N, 130°W) off the coast of Washington, with a strike direction of 107° east of true north; these details will become important shortly. The profile in Figure 8.1 shows a short section of 100 points taken out a much longer profile; the age of the

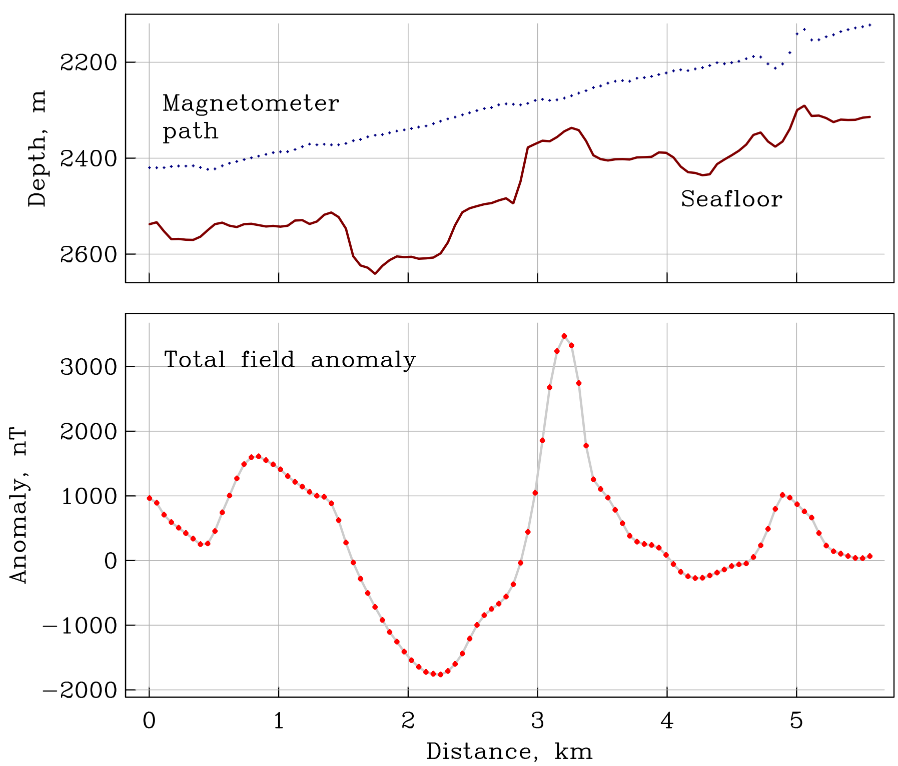


Figure 8.1: Magnetic anomaly and fish track.

oceanic crust is about 0.2 Ma, so we are in the Bruhnes normal chron, which ended 0.78 Ma ago.

In a moment we are going to make a number of simplifications to get at the crustal magnetization here. Recall from 2(1-2) that the original magnetization forward problem has a solution that looks like this:

$$\Delta\mathbf{B}(r) = \int_V \mathbf{G}(\mathbf{s},\mathbf{r}) \cdot \mathbf{M}(\mathbf{s}) \, d^3\mathbf{s} \tag{1}$$

with **G** fully expanded now to

$$\mathbf{G}(\mathbf{s},\,\mathbf{r}) = \frac{\mu_0}{4\pi}\,\hat{\mathbf{B}}_0 \cdot \nabla\nabla\,\frac{1}{R} = \frac{\mu_0}{4\pi}\left[\frac{3\hat{\mathbf{B}}_0 \cdot (\mathbf{r}-\mathbf{s})(\mathbf{r}-\mathbf{s})}{R^5} - \frac{\hat{\mathbf{B}}_0}{R^3}\right] \tag{2}$$

and $R = |\mathbf{r}-\mathbf{s}|$. Remember the magnetization **M** is a vector-valued function of position **s** within the volume V; the grad acts on the **s** coordinate here, although since $G(\mathbf{r},\mathbf{s}) = G(\mathbf{s},\mathbf{r})$ that is unimportant in (2). In (1) **G** is known and **M** is the function we seek. In fact we do not have continuous values of $\Delta\mathbf{B}$, but samples at specific points on the sea surface $\mathbf{r}_1, \mathbf{r}_2, \cdots \mathbf{r}_m$. So we can specialize (1) to this situation

$$d_j = \Delta\mathbf{B}(\mathbf{r}_j) = \int_V \mathbf{G}_j(\mathbf{s}) \cdot \mathbf{M}(\mathbf{s}) \, d^3\mathbf{s}, \quad j = 1, 2, \cdots m \tag{3}$$

where we assign particular function $G_j(\mathbf{s})$ to each observation:

$$\mathbf{G}_j(\mathbf{s}) = \mathbf{G}(\mathbf{s},\mathbf{r}_j) = \frac{\mu_0}{4\pi}\,\hat{\mathbf{B}}_0 \cdot \nabla\nabla\,\frac{1}{|\mathbf{r}_j - \mathbf{s}|} \tag{4}$$

Notice that each measured number d_j is obtained in (3) via a **linear functional** of the unknown **M**. The set of vector-valued magnetizations within the volume V can be considered to be linear vector space $\mathcal{V}$; there are obvious rules for adding two magnetizations and for multiplying a given magnetization by a scalar. Suppose we equip $\mathcal{V}$ with the following **inner product**:

$$(\mathbf{f},\,\mathbf{g}) = \int_V \mathbf{f}(\mathbf{s}) \cdot \mathbf{g}(\mathbf{s}) \, d^3\mathbf{s} \tag{5}$$

The we automatically get a **norm** for the space; the new normed space will be called $\mathcal{H}$ for Hilbert. The implied size of a given magnetization function is this:

$$\|\mathbf{M}\| = \sqrt{\int_V \mathbf{M} \cdot \mathbf{M} \, d^3 s} \tag{6}$$

This is almost the RMS magnetization; to get $\|\mathbf{M}\|$ to be RMS magnetization we have to normalize by the volume: $M_{\text{RMS}} = \|\mathbf{M}\|/V^{1/2}$.

The key question is whether or not we can write (3) as an inner product or not. It is a linear functional as we already stated, but is it a **bounded** linear functional? Suppose I write (3) as an inner product:

$$d_j = (\mathbf{G}_j, \mathbf{M}) \tag{7}$$

Then (7) is a valid inner product if and only if $||G_j|| < \infty$, that is $\mathbf{G}_j$ is a proper element of $\mathcal{H}$, the normed space. In this case that is easy to verify. Equation (2) looks complicated, but $\mathbf{r}$ and $\mathbf{s}$ never become identical, because the magnetometer is at the ocean surface $\mathbf{r}_j$ and the sources at $\mathbf{s}$ are under water. This means the function $\mathbf{G}_j(\mathbf{s})$ never becomes infinite anywhere in V; and the volume V is finite too. Hence $\mathbf{G}_j \in \mathcal{H}$. In this way have put the seamount magnetization problem in a Hilbert space setting, writing the solution to the forward problem in a natural manner as an inner product, and using essentially RMS magnetization as a norm. I must add that it would be possible to choose other forms for norm and inner product here, (based on gradients, to smooth things out, for example) but they add complications, best avoided for now.

The shape of seamounts is complex and the computational problem rather large so we return to the profile I plotted earlier from the Juan de Fuca Rise for another version of the magnetization inverse problem. To simplify things we are going to assume next: (i) No variation of magnetization into the plane of the profile, the y direction; (ii) Direction of magnetization constant, $\hat{\mathbf{M}}_0$; (ii) The magnetized layer has a constant thickness Δz ; (iii) Strength of magnetization varies only with x the horizontal coordinate, and not with z within the layer. Then (1) becomes an integral of x alone involving the magnetic intensity $m(x)$; we evaluate the j-th measurement made at the point $\mathbf{r}_j$:

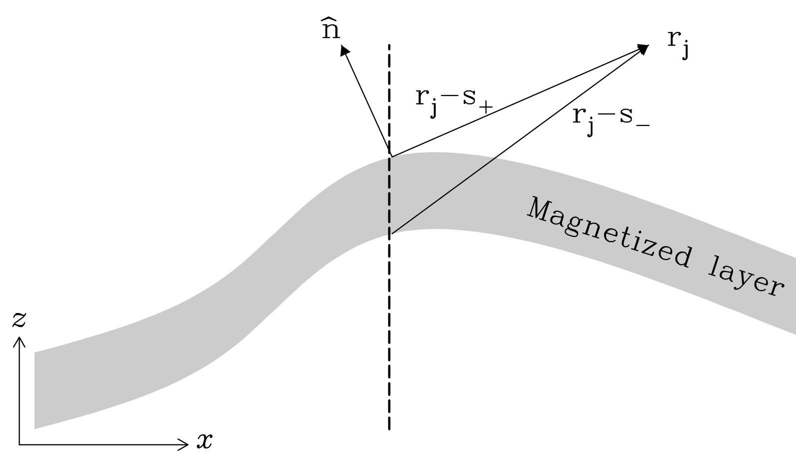


Figure 8.2: Model of magnetic layer.

$$d_j = \Delta \mathbf{B}(\mathbf{r}_j) = \int g(\mathbf{r}_j, x)\, m(x)\, dx \tag{8}$$

$$g(\mathbf{r}_j, x) = \frac{\mu_0}{2\pi}\left[\frac{\hat{\mathbf{M}}_0 \cdot (\mathbf{r}_j - \mathbf{s}_+(x))}{|\mathbf{r}_j - \mathbf{s}_+(x)|^2} - \frac{\hat{\mathbf{M}}_0 \cdot (\mathbf{r}_j - \mathbf{s}_-(x))}{|\mathbf{r}_j - \mathbf{s}_-(x)|^2}\right]\hat{\mathbf{n}}(x) \cdot \hat{\mathbf{B}}_0 \tag{9}$$

where $\mathbf{s}_+$ and $\mathbf{s}_-$ are vectors in the plane pointing to the top and bottom of a vertical column of magnetic material (see Figure 8.2), $\hat{\mathbf{n}}$ is the surface normal at the top of the column. To get this equation we have integrated (2) both vertically through the layer and from $-\infty$ to ∞ in the y direction. For the data shown in Figure 8.1, because the crust is young we can assume $\hat{\mathbf{M}}_0 = \hat{\mathbf{B}}_0$, the ambient field direction at the site; both these vectors must be projected onto the observation plane. Then, after finding the inclination and dip at the Rise in 1995, (57.42°, 15.42°), we calculate that $\hat{\mathbf{B}}_0 = (0.018, -0.99986)$ in the plane of the profile, essentially vertically downward. We will use the approximation that $\hat{\mathbf{M}}_0 = \hat{\mathbf{B}}_0 = \hat{\mathbf{z}}$ from now on. This problem is interesting and we may return to it later in this form. But to get a magnetization inverse problem so simple we can solve all the integrals analytically we need even more drastic simplification.

We make two further approximations that will be relaxed later: first, we will take the track and the surface of the basement to be horizontal, flat lines. This looks like a serious error from Figure 8.1, but there is a factor of 15:1 in the plot of the track and bathymetry. And finally, we will take the layer thickness, Δz to be small, so that the magnetization can be treated as a thin sheet of dipoles. This assumption is highly suspect: magnetic layer thicknesses at oceanic rises are thought to be at least 500 m. With these further approximations (8) and (9) become:

$$d_j = \int g_j(x)\, m(x)\, dx \tag{10}$$

with

$$g_j(x) = \frac{\mu_0 \Delta z}{2\pi}\, \frac{h^2 - (x - x_j)^2}{(h^2 + (x - x_j)^2)^2} \tag{11}$$

Here the height above the basement is the constant $h = 174$ meters on average; measurements of the anomaly are taken at the horizontal coordinates x_j. This is the result used in GIT 2.06(2), with the addition of the factor Δz, omitted there by oversight, because Δz = 1 km in all the calculations!

We have to decide on an interval for the integration. In the idealized problem we will pretend the magnetic layer extends over the whole real line. This gross idealization does not get us into trouble until a bit later, and is very handy for solving the integrals. If we like the norm

$$\| m \| = \left(\int_{-\infty}^{\infty} m(x)^2\, dx \right)^{1/2} \tag{12}$$

then the space of magnetization models becomes the classic Hilbert space $L_2(-\infty, \infty)$, with inner product

$$(f, g) = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx \tag{13}$$

Is (11) an inner product in this space? The answer is yes if we can show that

$$\| g_j \|^2 = \frac{\mu_0^2 \Delta z^2}{4\pi^2} \int_{-\infty}^{\infty} \frac{(h^2 - (x - x_j)^2)^2}{(h^2 + (x - x_j)^2)^4}\, dz \tag{14}$$

is finite. Later we will evaluate this messy integral exactly. For now we can show it is bounded, by a couple of simple observations: the function $g_j(x)^2$ is bounded and continuous (in fact it is analytic on the real line); as $|x| \to \infty$ we can easily verify that $g(x) \to \mu_0 \Delta z / 2\pi x^2$ so that $g(x)^2 \to$ constant$/x^4$. This dies away fast enough to have a finite integral and there we can write (10) as

$$d_j = (g_j, m), \quad j = 1, 2, \cdots m \tag{15}$$

Again, this may not be the only norm we will want to use, but to get things started $L_2(-\infty, \infty)$ is a good place to start.

You may be asking, when would the linear functions fail to be bounded, and what would be the consequences of that failure? In this problem, we can cause the integrals like (14) to be undefined simply by setting h the height of the observation line, go to zero, effectively setting the magnetometer directly on the sources, instead of above them; this is not an impossible experimental geometry. One way to look at the failure is to examine the limit as h tends to zero: we find the norm minimizing magnetization becomes more and more closely concentrated around the observation sites, and has a smaller and smaller norm, in the limit, $\| m \| = 0$. The model space L_2 is inappropriate because it does not lead to a geophysically plausible, let alone a simple solution. The original justification for norm minimizers was that they should select models with the fewest extraneous features, to be as bland and unobjectionable as possible. When h is set to zero, L_2 fails to do this, and we should look for a different approach; the best way is to leave Hilbert space entirely.

9. The Continuous Problem

Before getting into the application of our general Hilbert-space based approach, let us look at a naive treatment of the 1-D, flat layer problem that relies on its special characteristics. While the method in its bare form fails miserably, it provides some useful insights of a general nature and, as we shall see later, it can be modified to yield quite satisfactory answers. First pretend that we have dense observations on the whole real line. Then (8.10) becomes

$$d(y) = \int_{-\infty}^{\infty} k(y-x)\, m(x)\, dx \qquad (1)$$

where

$$k(x) = \frac{\mu_0 \Delta z}{2\pi} \frac{h^2 - x^2}{(h^2+x^2)^2} \qquad (2)$$

Written this way, it is clear that the solution to the forward problem when the data d are known on $(-\infty, \infty)$ is a *convolution*.

Convolutions are very common in geophysics, and just as in this case, we frequently want to undo them: this is called the **deconvolution** problem. Geophysical examples include undoing the effects of an instrument, like a seismometer, or a magnetometer (Constable and Parker, 1991), or removing path effects from a seismogram using the empirical Green's function (Prieto, et al, 2006).

Whenever you see a convolution (which you know can be written $d = k * m$) you should think of the **Convolution Theorem**, which says

$$\mathcal{F}[k * m] = \mathcal{F}[k]\, \mathcal{F}[m] = \hat{k}\, \hat{m} \qquad (3)$$

where of course $\mathcal{F}$ is the Fourier transform (FT) given by

$$\mathcal{F}[f] = \hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \lambda x}\, dx \qquad (4)$$

Thus, if we take the FT of (1) we see

$$\hat{d} = \hat{k}\, \hat{m} \qquad (5)$$

and the solution to the inverse problem is obtained merely by division, since the complicated integral in (1) has been converted into a simple multiplication. (Mathematicians would say the convolution has been *diagonalized*.) The result for m is just

$$m(x) = \mathcal{F}^{-1}\left[\frac{\hat{d}}{\hat{k}}\right] \qquad (6)$$

The inverse problem is apparently finished, since the solution has been reduced to three FTs and a division.

In fact we can go one step further: the FT of $k(x)$ can be found in terms of elementary functions. Those of you who took SIO-239 from me should think about how to go about this calculation, starting from the well-known FT of $1/(h^2 + x^2)$. I will just give you the answer, the amazingly simple:

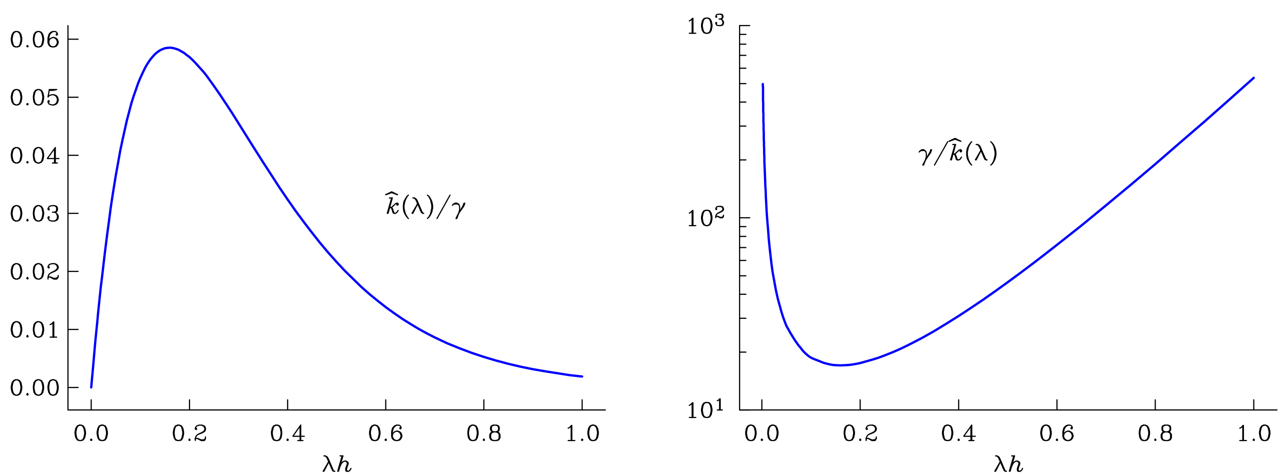
$$\hat{k}(\lambda) = \pi \mu_0 \Delta z \ |\lambda| \ \mathrm{e}^{-2\pi h |\lambda|} \tag{7}$$

This function is shown in Figure 9.1; it is obviously even in λ.

At first glance, equation (6) seems to imply there is a unique solution to the inverse problem: for every function d, there is a unique $\hat{d}$ and so there is just one $\hat{m}$ and one m that results: in other words, complete and exact data provide a single, one-dimensional magnetization model $m(x)$. That is in complete contrast to the 3-D problem considered at the beginning of the class, where exact data can be matched with a enormous array of different internal magnetizations. However, things aren't quite so straightforward. Look at (7): notice that $\hat{k}(0) = 0$. This means that the integral of $k(x)$ over all x vanishes, and therefore any constant magnetization generates zero magnetic anomaly. A constant magnetization may be present in any amount and it will always be undetectable: it is the **magnetic annihilator** for the problem. Hence the solution to this inverse problem is not unique — there is a single uncontrolled degree of freedom. We conclude that the many additional assumptions and simplifications we introduced have almost, but not quite, suppressed the vast ambiguity of the original problem posed in Section 2 of these notes.

The analytic solution also highlights the instability associated with short-wavelength anomalies. Equation (6) states that to obtain the magnetization function we must apply a spatial filter with transfer function $\hat{k}(\lambda)^{-1}$, a function shown in Figure 9.1. Because $\hat{k}(\lambda)$ decays essentially exponentially, small-scale components of d are magnified with an exponentially growing factor as the scale decreases. With realistic signals, noise does not decay exponentially with wavenumber, and so above some value of λ, noise will be preferentially amplified by this deconvolution.

Figure 9.1: The function $\hat{k}(\lambda)$ and its reciprocal. The constant $\gamma = \pi \mu_0 \Delta z / h$.



And the same thing happens, though less explosively at small wavenumbers. So both the longest and the short wavelengths are amplified in a singular way, making them the least reliable parts of the solution.

Applying (6), an "exact" solution, to real data presents serious difficulties. We don't have a continuous function $d(x)$ for all real x, but finite set of numbers. If we interpolate the samples to create values at short scales, we are introducing values just where the filter $1/\hat{k}$ generates the greatest exaggeration. If we extend the finite profile by some other kind of extrapolation, the longest wavelengths are invented, and yet these are given the unbounded emphasis by the filter! So, while we could attempt to approximate the analytic inversion with a Fast Fourier Transform of the actual data, we will turn instead to a process that takes into account the finiteness of the data set, and its uncertainties.

References

Constable, C. G. and Parker, R. L., Deconvolution of long-core paleomagnetic measurements: Spline therapy for the linear problem, *Geophys. J. Int.* 104, 453-468, 1991.

Prieto, G. A., Parker, R. L., Vernon, F. L., Shearer, P., M., and Thomson, D. J., Uncertainties in earthquake source spectrum estimation using empirical Green functions, in *Earthquakes: Radiated Energy and the Physics of Faulting,* Geophysical monograph 170, eds Abercrombie, McGarr, Kanamori, and Di Toro, American Geophysical Union, Washington, 2006.

10. The Minimum Norm Solution

Staying with the very idealized 1-D problem in a flat, thin layer, we address directly the question of a finite data set using the Hilbert space theory. In the previous Section we found that trying to complete the set of observations to make them suitable for the exact solution automatically introduced artificial values exactly where the wavenumber filter put the most emphasis, the longest and shortest wavelengths. To get around this problem we acknowledge from the start the finite of the data set by writing

$$d_j = \int_{-\infty}^{\infty} g_j(x)\, m(x)\, dx. \quad j = 1, 2, \cdots N \tag{1}$$

Notice that the model is still defined as a real function on $(-\infty, \infty)$. However, the price to be paid is the fact that there cannot now be a unique solution, since only finitely many conditions result from (1), but a function is needed. So we invoke the principle of minimum complexity, and look only for models that are small in some sense to avoid the exponential magnifications implied by the deconvolution study. The sense will be the norm, and for simplicity of calculation we shall say that $m \in L_2(-\infty, \infty)$. Therefore we write (1) as a set of inner products in L_2:

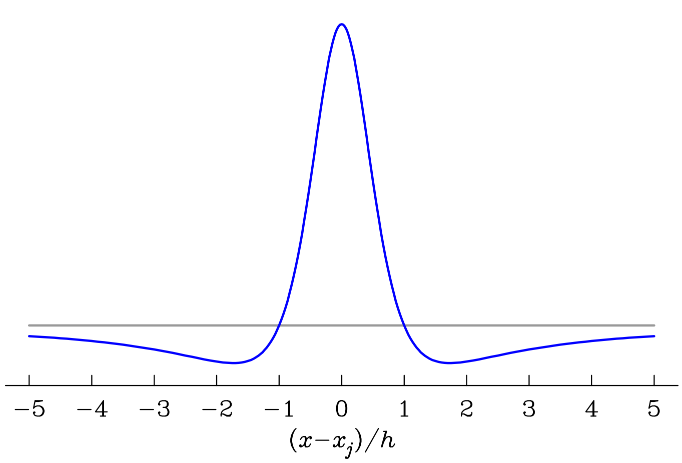
$$d_j = (g_j, m), \quad j = 1, 2, \cdots N \tag{2}$$

where from (8.11)

$$g_j(x) = \frac{\mu_0 \Delta z}{2\pi} \frac{h^2 - (x - x_j)^2}{(h^2 + (x - x_j)^2)^2} \tag{3}$$

These N elements of L_2 are the **representers** for the problem, although it has not yet been proved that the linear functionals in (2) are in fact bounded (they are), or equivalently that $\| g_j \|$ exists. Figure 10.1 depicts a typical representer; they are all the same shape, just shifted in x because (1) is a convolution.

Figure 10.1: A typical representer for the simplified magnetic problem.



Recall the result from GIT that the element that achieves the smallest norm while satisfying (2) is the linear combination of representers:

$$m_0 = \sum_{j=1}^{N} \alpha_j g_j \tag{4}$$

The coefficients α_j are found by substituting this expansion back into (2) and solving the linear system

$$\sum_{k=1}^{N} \Gamma_{jk} \alpha_k = d_j, \quad j = 1, 2, \cdots N \tag{5}$$

where Γ_{jk} are the entries of the **Gram matrix** of representers:

$$\Gamma_{jk} = (g_j, g_k), \quad j = 1, 2, .. N; \; k = 1, 2, .. N \tag{6}$$

It is shown in GIT that if the representers are linearly independent, Γ is nonsingular and so (6) has a unique solution. We will prove that at the end of this Section.

To make progress we must evaluate the Gram matrix; the integrals seem at first sight to be quite daunting. But if we recall the result that an inner product is invariant under Fourier transformation:

$$(g_j, g_k) = (\hat{g}_j, \hat{g}_j) \tag{7}$$

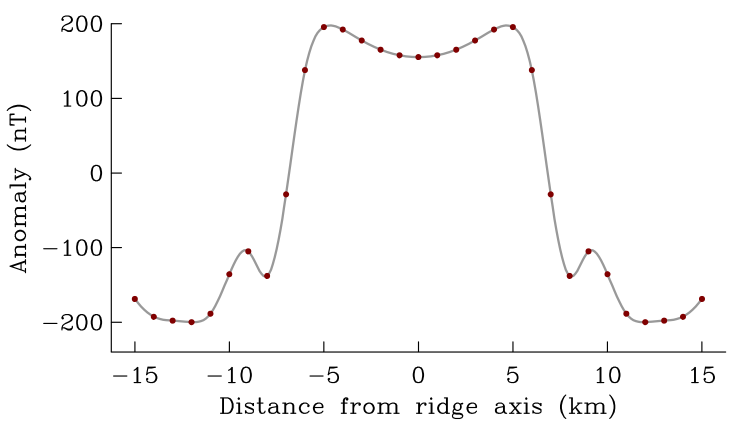
the task becomes more tractable, especially in view of the simplicity of the FT in (8.7). We find after a modest amount of work (see GIT, p 80) that

$$\Gamma_{jk} = \frac{\mu_0^2 \Delta z^2 h}{\pi} \, \frac{4h^2 - 3(x_j - x_k)^2}{[4h^2 + (x_j - x_k)^2]^3} \tag{8}$$

Since $\Gamma_{jj} = \| g_j \|^2$, this result proves the norms of the representers are finite, and the integrals in (1) are bounded linear functionals as we have assumed.

Before applying an inversion to real data it is wise to test it on an artificial set generated from a known model. So that is what we do next, the data set is drawn from Chapter 2 of GIT and is shown below. Here the scale is very different from the near-bottom data shown in Section 8.

Figure 10.2: Fake marine magnetic anomaly data.

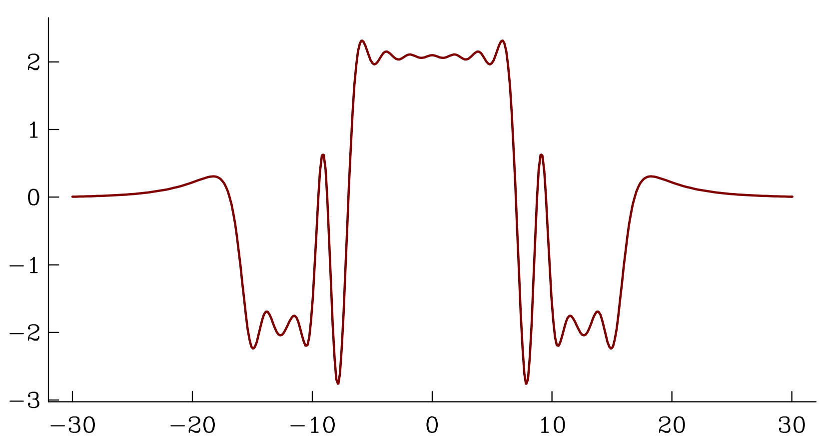


the water depth $h = 2\,\text{km}$ and the layer thickness $\Delta z = 1\,\text{km}$. The minimum L_2 norm solution is obtained by a few lines of MATLAB:

```
h=2;  dz=1;  muo=4*pi*100;
c = muo^2*dz^2*h/pi;

x = [-15 : 15]';
d =[-168.72,-192.57,-197.78,-199.79,-188.49,-135.62,-104.91,-137.87, ...
-28.557,138.00,195.60,192.36,177.66,165.33,157.87,155.41,157.87, ...
165.33,177.66,192.36,195.60,138.00,-28.557,-137.87,-104.91, ...
-135.62,-188.49,-199.79,-197.78,-192.57,-168.72]';

%  Build Gram matrix
[X Y]=meshgrid(x,x);
G=c*(4*h^2-3*(X-Y).^2)./(4*h^2+(X-Y).^2).^3;

%  Solve linear system
a=G\d;

%  Represents are columns of g
y=linspace(-30,30,200);
[X Y]=meshgrid(x,y);
g=(muo*dz/(2*pi))*(h^2-(X-Y).^2)./(h^2+(X-Y).^2).^2;

%  Solution is a linear combo of representers
mo = g*a;
```

The norm minimizer, m_0, is plotted below. You can guess what the original model was that generated the data from this solution. The result is fairly wiggly, though it has small amplitude as it should. We should now think about suppressing these wiggles, say by finding the solution that minimizes $\| dm/dx \|$. This is discussed in GIT, using a pure Hilbert space method. We will look at this from a numerical view point in a later

Figure 10.3: L_2 norm minimizing magnetization from artificial data shown in Figure 10.2.



Section. The results are not very impressive — the wiggles are still there.

In GIT the linear independence of the representers for the 1-D magnetic anomaly inverse problem was proved using the Fourier transform. I will present here an alternative proof, which is more useful because it gives a method that I have found works for a number of linear inverse problems, especially those arising out of potential theory like this one. The approach is successful for problems in two and three dimensional problems too.

Here we have a set of N functions on the real line given by:

$$g_j(x) = \frac{h^2 - (x - x_j)^2}{(h^2 + (x - x_j)^2)^2}\,, \quad j = 1, 2, \cdots N \tag{9}$$

where all the x_j are different (why?). I have dropped all the constant factors for obvious reasons. Although the setting of the original problem is the space $L_2(-\infty, \infty)$, we can use properties of g_j that functions in L_2 don't all possess for our proof. The normal way to prove this kind of proposition is by contradiction: we shall assume the set is **linearly dependent** and then show that leads to a contradiction. The statement for LI is that there are scalars β_j not all zero, such that

$$\sum_{j=1}^{N} \beta_j g_j(x) = 0, \text{ for all } x\,. \tag{10}$$

If this is true, we can choose one of the functions g_k multiplied by a nonzero coefficient, and move it onto the other size thus:

$$g_k(x) = \sum_{j \neq k} (-\beta_j/\beta_k)\, g_j(x) \tag{11}$$

which says that the representer g_k can be built from a linear combination of the others.

To get a contradiction we need to focus on a value of x where $g_k(x)$ goes to infinity. There is no such place on the real line, but never mind — we can choose x to be **complex** if we wish! Looking at (9) we see that the denominator vanishes whenever $(x - x_j)^2 = -h^2$ or

$$x_* = x_j \pm \mathrm{i}h\,. \tag{12}$$

These are the singularities of g_j in the complex x plane. Suppose now I consider the function $g_k(x)$ in (11) when $x - x_k = \mathrm{i}h + \varepsilon$ where ε is a small real quantity of our choosing; in other words, x is very near a singular point of the function $g_k(x)$. From the definition of g_k we see that

$$g_k(x) = \frac{h^2 - (\mathrm{i}h + \varepsilon)^2}{(h^2 + (\mathrm{i}h + \varepsilon)^2)^2} = -\frac{2h^2 + 2\mathrm{i}h\varepsilon + \varepsilon^2}{\varepsilon^2(2\mathrm{i}h + \varepsilon)^2} \tag{13}$$

$$= \frac{1}{2\varepsilon^2}\left[1 + O(\varepsilon)\right]. \tag{14}$$

It is clear that by making ε small enough we can get the magnitude of

$g_k(x)$ as large as we wish. For the same x on the other side of (11) we look at the denominators and we see a value that does not go to zero as ε gets small:

$$(h^2+(x-x_j)^2)^2 = ((x_k - x_j + \varepsilon)^2 + 2\mathrm{i}h(x_k - x_j + \varepsilon))^2 \tag{15}$$

$$= (x_k - x_j + \varepsilon)^2 (x_k - x_j + \varepsilon + 2\mathrm{i}h)^2 \,. \tag{16}$$

This number is not zero for small ε because $x_j \neq x_k$ and both of the xs are real. The same thing is true for every j on the right of (11). This means that as ε tends to zero, the left size of (11) grows and grows, while the right side tends to some constant value. That is a contradiction, which means that constants like β_j do not exist.

Almost exactly this argument works for the representers in the more complicated magnetic problems considered in Section 8 of the Supplementary Notes, while the Fourier treatment probably could not be made to apply. Setting up an equation like (11) in which one side is constant, while the other grows to infinity (or shrinks to zero) is a classic way to show linear dependence of a set of functions. Doing this in the complex plane is a very natural thing, because every analytic function has a singularity in the complex plane, except the constant.