$$g(\mathbf{r}_j, x) = \frac{\mu_0}{2\pi}\, \hat{\mathbf{I}}(\hat{\mathbf{B}}_0, \hat{\mathbf{M}}_0) \cdot \left[\frac{\mathbf{r}_j - \mathbf{s}_+(x)}{|\,\mathbf{r}_j - \mathbf{s}_+(x)\,|^2} - \frac{\mathbf{r}_j - \mathbf{s}_-(x)}{|\,\mathbf{r}_j - \mathbf{s}_-(x)\,|^2} \right] \tag{9}$$

where $\hat{\mathbf{I}}$ is the unit vector given by

$$\hat{\mathbf{I}} = \begin{bmatrix} \sin(I_B + I_M) \\ -\cos(I_B + I_M) \end{bmatrix} \tag{9a}$$

and the angles I_B and I_M are the inclinations of the field and magnetization vectors in the x-z plane. Also $\mathbf{s}_+$ and $\mathbf{s}_-$ are vectors in the plane pointing to the top and bottom of a vertical column of magnetic material (see Figure 8.2). To get this equation we have integrated (2) both vertically through the layer and from $-\infty$ to ∞ in the y direction. For the data shown in Figure 8.1, because the crust is young we can assume $\hat{\mathbf{M}}_0 = \hat{\mathbf{B}}_0$, the ambient field direction at the site; both these vectors must be projected onto the observation plane. Then, after finding the inclination and dip at the Rise in 1995, (57.42°, 15.42°), we calculate that $\hat{\mathbf{B}}_0 = (0.018, -0.99986)$ in the plane of the profile, essentially vertically downward. We will use the approximation that $\hat{\mathbf{M}}_0 = \hat{\mathbf{B}}_0 = \hat{\mathbf{z}}$ from now on. This problem is interesting and we may return to it later in this form. But to get a magnetization inverse problem so simple we can solve all the integrals analytically we need even more drastic simplification.

We make two further approximations that will be relaxed later: first, we will take the track and the surface of the basement to be horizontal, flat lines. This looks like a serious error from Figure 8.1, but there is a factor of 15:1 in the plot of the track and bathymetry. And finally, we will take the layer thickness, Δz to be small, so that the magnetization can be treated as a thin sheet of dipoles. This assumption is highly suspect: magnetic layer thicknesses at oceanic rises are thought to be at least 500 m. With these further approximations (8) and (9) become:

$$d_j = \int g_j(x)\, m(x)\, dx \tag{10}$$

with

$$g_j(x) = \frac{\mu_0 \Delta z}{2\pi} \frac{h^2 - (x - x_j)^2}{(h^2 + (x - x_j)^2)^2} \tag{11}$$

Here the height above the basement is the constant $h = 174$ meters on average; measurements of the anomaly are taken at the horizontal coordinates x_j. This is the result used in GIT 2.06(2), with the addition of the factor Δz, omitted there by oversight, because Δz = 1 km in all the calculations!

We have to decide on an interval for the integration. In the idealized problem we will pretend the magnetic layer extends over the whole real line. This gross idealization does not get us into trouble until a bit later, and is very handy for solving the integrals. If we like the norm

$$\| m \| = \left(\int_{-\infty}^{\infty} m(x)^2 \, dx \right)^{1/2} \tag{12}$$

then the space of magnetization models becomes the classic Hilbert space $L_2(-\infty, \infty)$, with inner product

$$(f, g) = \int_{-\infty}^{\infty} f(x) \, g(x) \, dx \tag{13}$$

Is (11) an inner product in this space? The answer is yes if we can show that

$$\| g_j \|^2 = \frac{\mu_0^2 \Delta z^2}{4\pi^2} \int_{-\infty}^{\infty} \frac{(h^2 - (x - x_j)^2)^2}{(h^2 + (x - x_j)^2)^4} \, dz \tag{14}$$

is finite. Later we will evaluate this messy integral exactly. For now we can show it is bounded, by a couple of simple observations: the function $g_j(x)^2$ is bounded and continuous (in fact it is analytic on the real line); as $|x| \to \infty$ we can easily verify that $g(x) \to \mu_0 \Delta z / 2\pi x^2$ so that $g(x)^2 \to$ constant $/x^4$. This dies away fast enough to have a finite integral and there we can write (10) as

$$d_j = (g_j, m), \;\; j = 1, 2, \cdots m \tag{15}$$

Again, this may not be the only norm we will want to use, but to get things started $L_2(-\infty, \infty)$ is a good place to start.

You may be asking, when would the linear functions fail to be bounded, and what would be the consequences of that failure? In this problem, we can cause the integrals like (14) to be undefined simply by setting h the height of the observation line, go to zero, effectively setting the magnetometer directly on the sources, instead of above them; this is not an impossible experimental geometry. One way to look at the failure is to examine the limit as h tends to zero: we find the norm minimizing magnetization becomes more and more closely concentrated around the observation sites, and has a smaller and smaller norm, in the limit, $\| m \| = 0$. The model space L_2 is inappropriate because it does not lead to a geophysically plausible, let alone a simple solution. The original justification for norm minimizers was that they should select models with the fewest extraneous features, to be as bland and unobjectionable as possible. When h is set to zero, L_2 fails to do this, and we should look for a different approach; the best way is to leave Hilbert space entirely.