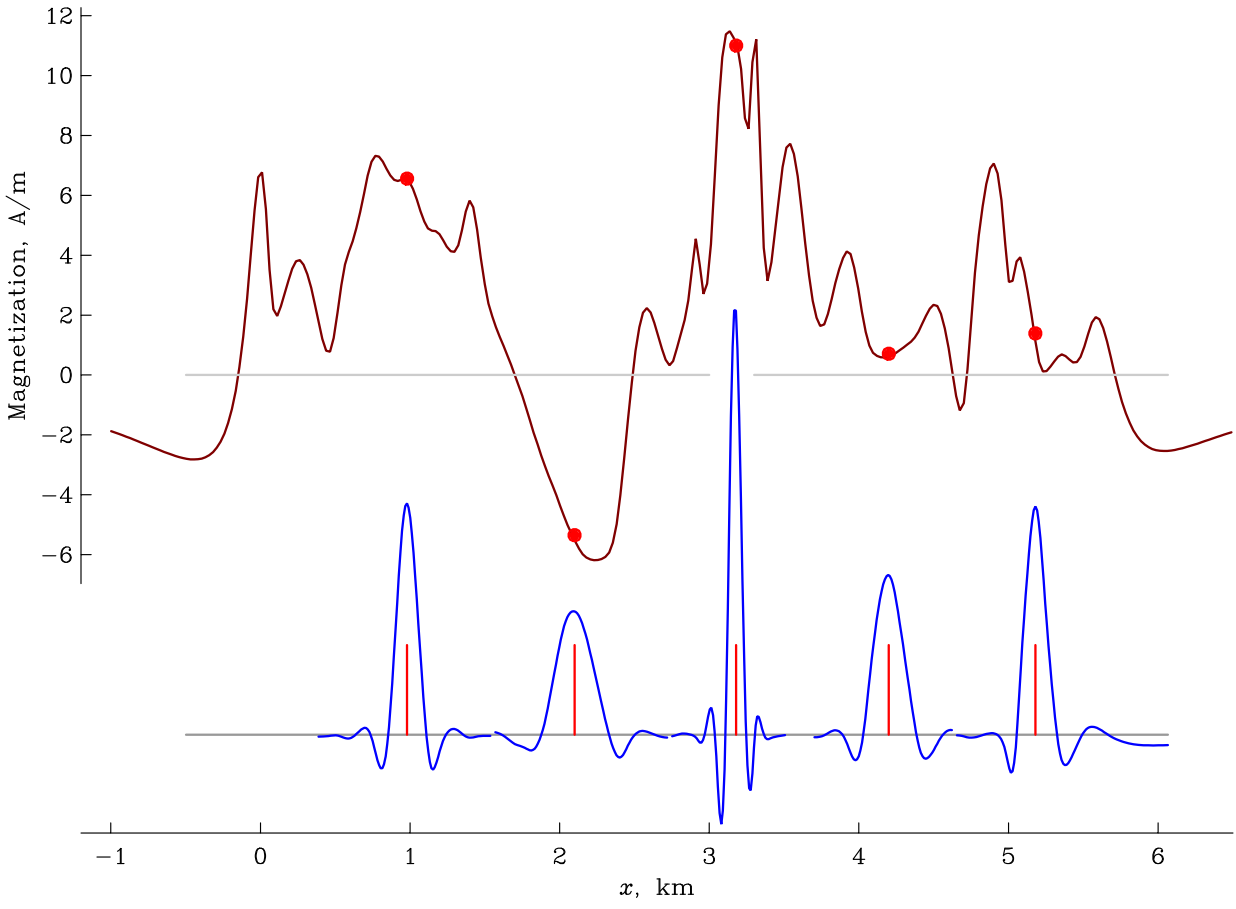
14. Resolution in the Marine Magnetics Example

We carry out the resolution analysis for the model on the Juan de Fuca Ridge that has been our standard vehicle in the linear theory. As explained in class, one of the fairly traditional ways of doing this is to calibrate the regularization process by testing it with artificial data, generated from special models, often delta functions, at various points. This must be done using exactly the same parameter settings as were used in making the regularized model. Then we see at each point how badly smeared out a very sharp feature would be after being processed through the inversion machinery. This gives us a qualitative assessment of the length scales resolved in the solution.

Figure 14.1 shows the results for five locations of the test function. We see the resolution varies by quite a large factor from place to place, but is reasonably satisfactory everywhere. At $x = 2\,\text{km}$ we may be resolving structure only down to about 0.5 km, while near 3 km the resolution improves to almost 0.1 km. The cause of the variation is easy to discover in this case: resolving scale is apparently proportional to the distance to the nearest source material, as you can easily see by comparing this figure with Figure 8.1.

Figure 14.1: Top: L_2-norm minimizing magnetization. Bottom: Resolution functions for various sites.



15. Practical Calculation of a Bound

In Section 4.03 of GIT I treat the question of finding the range of a linear functional subject to constraints from data (linear functions) and a nonlinear conditions, that the norm be restricted as well. There in the noise-free case an elegant theory is developed for the noise-free case (data matched exactly by the model), and a number of predictions, also in the form of linear functionals. The problem with noise seems to be much more clumsy. Here I want to give a simpler for numerically formulated problems.

To motivate the discussion look again at Figure 14.1 and the negative magnetization near $x = 2\,\text{km}$. According to the resolution calculations, the regularized model should be trustworthy on the scale of a kilometer, and so we are inclined to accept the reality of reversed magnetization in this region. But is that truly required by the data? As GIT 4.03 shows us, we cannot be sure without introducing some further information. Suppose we say we know the L_2 norm of m. If we confine all magnetic material to $-0.5 \le x \le 6$ then $\| m \| / \sqrt{6.5} = M_{\text{RMS}}$, and we might be willing to specify an upper limit on plausible RMS magnetizations, from samples of fresh marine basalts. Then one way we might answer the question of a possible record of magnetic reversal is to consider the *average* value of magnetization over the x interval (x_1, x_2) where the regularized model dips negative; numerically this is (1.8, 2.4) km. We will call this $<m>$:

$$<m> = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} m(x)\, dx\,. \tag{1}$$

We challenge the hypothesis that the negative $<m>$ is required: if the every model that adequately matches the anomalies and has a *positive* average possesses an implausibly large RMS magnetization then a reversal is demanded in the interval. The converse, a positive average consistent with a reasonable RMS, does not mean the absence of a reversal however, because negative segments might still be needed, even with a positive average—it is just an average; we'll return to this later.

The idea then is to seek the smallest norm of the discretized model $m \in \mathbb{R}^N$:

$$\min_{m \in \mathbb{R}^N} \| Rm \| \tag{2}$$

subject to an adequate fit to the measurements:

$$\| \hat{d} - B\, m \| \le T \tag{3}$$

and the constraint on a linear functional of the model, in the example the average given in (1):

$$l^T m = b \tag{4}$$

where the scaled data $\hat{d}$ and matrix $B \in \mathbb{R}^{M \times N}$ are given in 13(8); $l \in \mathbb{R}^N$

is vector which approximates averaging the model over a region. If the smallest norm found in (2) is too large when (2) and (3) hold, we know that (3) cannot be supported. How does one apply the *equality* constraint (4)? Obviously one approach is introduce a second Lagrange multiplier, in addition to the one needed to account for (3). Let me describe a simpler approach. We treat (4) as another fictitious "observation" to be fitted (almost) exactly by including it in (3). Write a data vector $\hat{d}_1 \in \mathbb{R}^{N+1}$ and a new $B_1 \in \mathbb{R}^{(M+1)\times N}$ and a scalar γ:

$$B_1 = \begin{bmatrix} \gamma l^T \\ B \end{bmatrix}; \quad \hat{d}_1 = \begin{bmatrix} \gamma b \\ \hat{d} \end{bmatrix} \tag{5}$$

where $\gamma > 0$ is a very large constant. This clearly has the effect in the fitting problem of giving (4) a heavy weight, so that it will be fit more accurately than the rest of the data. Now we solve the standard norm minimization problem (2) subject to misfit (3) achieving a specified tolerance exactly as in (3), where B and $\hat{d}$ are replaced by B_1 and $\hat{d}_1$. But wait a minute—shouldn't we have to designate a new T as well? We have messed up the fitting process by including a very accurate fake datum. The answer is no: it can be shown (in Chapter 22, of *Solving Least Squares Problems,* by Lawson and Hanson, 1974) that if γ is large enough, the misfit component introduced by the new row in B_1 is negligible! How large is that? Lawson and Hanson give a very conservative figure; I find that if $\gamma \| l \| = 100 \| B \|$ we can get very good results. Too large a value for γ leads to numerical instability, unfortunately. The advantage of this approach is that we use exactly the same code when we bound the linear function l as when we find a regularized model.

We apply these ideas to the near-bottom magnetic anomalies. The linear functional we are interested in is the average value <m> in the interval where the regularized model goes negative, so if the length of that interval is $D = x_2 - x_1$ and the spacing in the discretized model is Δx, the vector l in (4) looks like this if we use the trapezoidal rule to approximate (1):

$$l = \frac{\Delta x}{D} [0, 0, \cdots 0, \tfrac{1}{2}, 1, 1, 1, 1, \cdots 1, \tfrac{1}{2}, 0, 0, \cdots 0]^T . \tag{6}$$

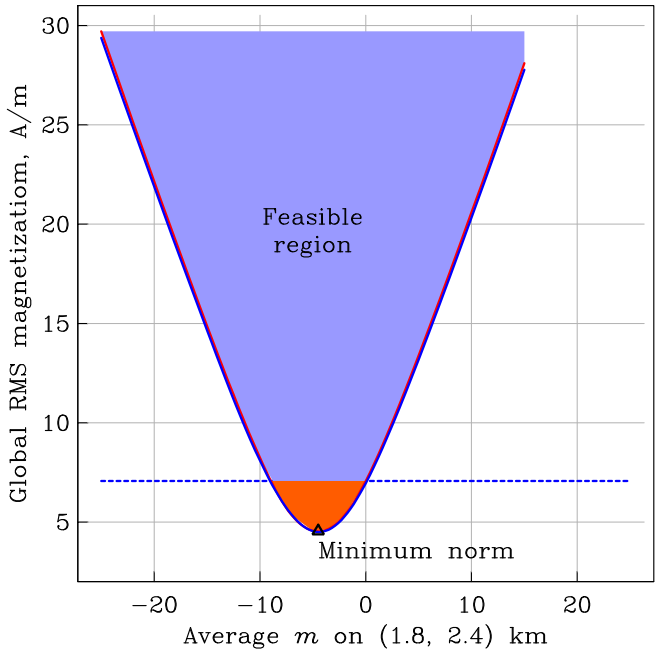
The first choice for b the bound in (4) might be zero, but in fact considerable insight is gained if we sweep b through a range of values. The results are plotted in Figure 15.1. For each value of <m> we plot the minimum possible norm, normalized to be RMS magnetization. That means

models exist with values above the point, but not below. Thus the curve of all such points is the boundary between models possessing the possible pairs of average-magnetization/RMS-magnetization and impossible pairs. So if we draw a horizontal line at some value of the RMS, the corresponding segment in the feasible zone gives us the largest and smallest <*m*> for that RMS, the bounds on the linear functional (4). So we have solved the problem for finding the upper and lower bounds on the linear functional. When we refer to the Figure we see that the value of <*m*> must certainly be negative if the RMS magnetization is less than 7.07 A/m, the orange region. Unfortunately, RMS magnetizations might easily be larger than this—it is not a very high value. The calculation is inconclusive. Note that this result *does not* back up the resolution analysis, which seems to say that we can trust the average value of the model in a 1-km interval. Apparently models with reasonable norms exist that have positive averages <*m*>.

What value of tolerance T should be use here? In fact we should use a larger T because we want to be really sure the misfit is not accidental. But as in earlier examples the choice makes hardly any difference: in the graph I have plotted the curves for expected value of T, and for a value that would not be exceeded in 95 percent of random realizations. The curves can only just be distinguished near the top of the graph; one is red, the other blue.

Let us look next at a model with a non-negative (i.e., slightly positive) mean value and reasonable norm, computed in the course of this

Figure 15.1: Minimum norm for specified value of the average magnetization <*m*>.



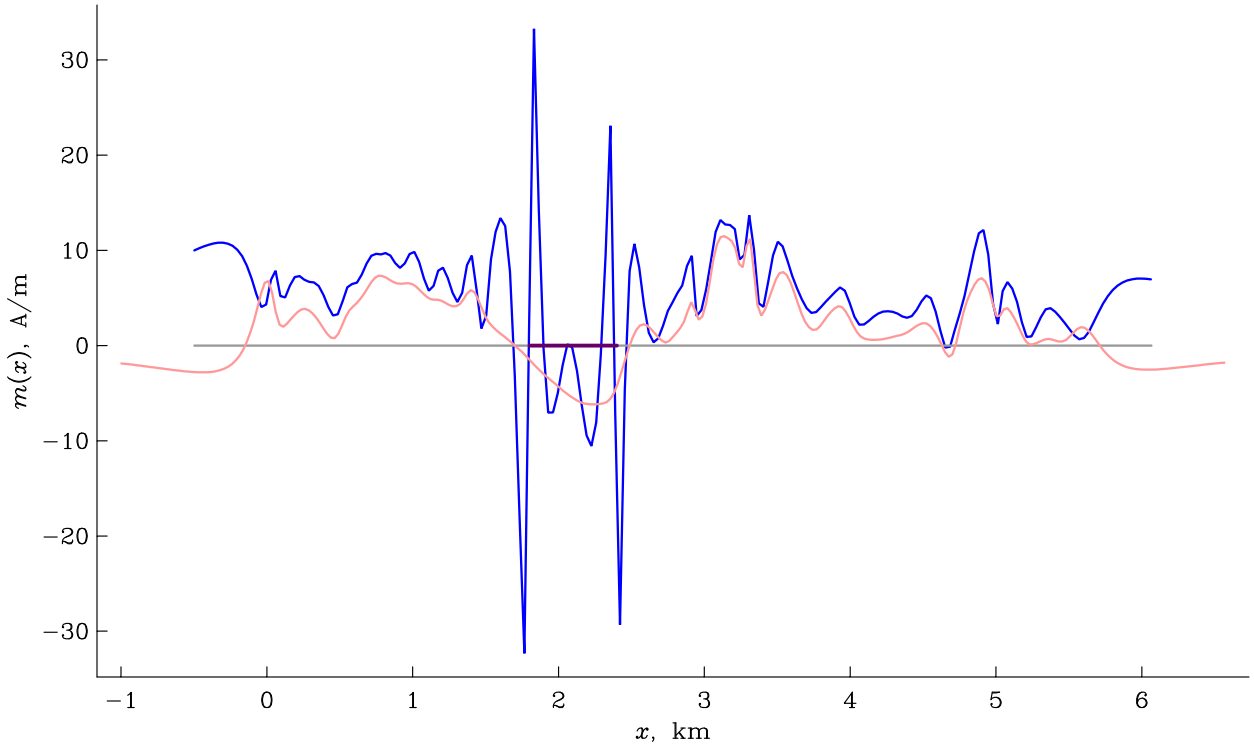


Figure 15.2: Model with reasonable RMS, fitting the data, and possessing a positive mean $<m> = 1$ A/m. The L_2 norm minimizer is shown light.

solution. The model with $<m> = 1$ A/m and RMS value 8.2 A/m is shown in Figure 15.2. Despite having an overall positive average over the interval of interest, the function *is still negative* in places, and a reversed magnetization is required. Furthermore, the model has large peak values, both negative and positive that make it seem unreasonable. Particularly troubling are the large negative swings, *just outside* the averaging interval. All of this strongly suggests that in this problem the use of the average value to test a hypothesis is not very effective. If we could test the hypothesis that any magnetization that is positive *everywhere* on the interval will not fit the data, then we might have a really strong test. That is the direction we are heading.

We have been pursuing the possibility that our magnetic data may demand a reversely magnetized section of crust, which would be quite interesting if it turned out to be true; in fact, because such a reversed segment has not been documented elsewhere, we want to be particularly certain it is required. So far, the evidence is not convincing. A reason peculiar to marine magnetic studies, is the very small response of the system to long wavelength magnetizations. Recall from Section 9, in the idealized case of a flat layer and horizontal observation track, a constant magnetization is completely invisible in the data: then $(g_j, c) = 0$ for

c = constant. In the more realistic model, this will not be true, but a long wavelength **annihilator** (a magnetization without a magnetic anomaly) is surely present. How can we find it? If one exists, it means the whole model m can be shifted up or down by arbitrary amounts without affecting the fit. This a problem we already know how to solve: if u the uniform (that is constant) magnetization, we seek $n \in \mathcal{H}$ as close as possible to u while satisfying $(g_j, n) = 0$. When closeness is measured in the usual way by the norm, we have the problem of finding a model near to a preferred structure, subject to constraints from data, here all zero in value. In symbols

$$n = \underset{m \in \mathcal{H}}{\text{argmin}} \; \| m - u \|, \quad \text{with} \;\; (g_j, m) = 0, \;\; j = 1, 2, .. N \tag{7}$$

The solution of this problem is one we have looked at briefly early on in class and in GIT (p 73). The result is rather too irregular, and so we weaken the demand requiring exactly zero magnetic signal, to that of an RMS signal of 0.73 nT. The result is shown below,. The solution is rather irregular, and so we weaken the demand from requiring exactly zero magnetic signal, to having an RMS signal of 10 nT. The result is shown below. While the function is far from constant, it is certain;y positive, and can be added to any solution with a weight of up to 20 before reaching a significant misfit.

Figure 15.3: Annihilator approximating a constant 1 A/m with RMS misfit 0.73 nT.