5. The DFT and Approximation with the FFT

We come full circle back to the original idea of an inner product as the natural generalization of the vector dot product. Consider the finite-dimensional vector space $\mathbb{C}^N$, the set of all complex N-vectors $[z_0, z_1, \cdots z_{N-1}]$. Notice the peculiar indexing scheme, which is used only for the particular case we are going to investigate. Now we equip this space with the inner product

$$(u, v) = \sum_{n=0}^{N-1} u_n v_n^* \,. \tag{5.1}$$

Then of course there is a norm here given by

$$\| u \| = (u, u)^{1/2} \tag{5.2}$$

and the idea of orthogonality between two vectors is obvious. Suppose we wish to expand an arbitrary complex vector z in an orthogonal basis, $e_0, e_1, \cdots e_{N-1}$:

$$z = \sum_{n=0}^{N-1} c_n e_n \tag{5.3}$$

then we know that the expansion coefficients are given by the familiar

$$c_n = (z, e_n)/\| e_n \|^2, \;\; n = 0, 1, \cdots N-1 \tag{5.4}$$

I will study a particular basis, one that looks suspiciously familiar from section 3. Suppose

$$e_n = [1, \mathrm{e}^{2\pi in/N}, \mathrm{e}^{4\pi in/N}, \cdots \mathrm{e}^{2k\pi in/N}, \cdots \mathrm{e}^{2(N-1)\pi in/N}] \,. \tag{5.5}$$

Each basis vector is in the form

$$e_n = [1, \omega_n, \omega_n^2, \omega_n^3, \cdots \omega_n^{N-1}] \tag{5.6}$$

where we write $\omega_n = \mathrm{e}^{2\pi in/N}$. This basis is orthogonal. To prove that fact consider

$$(e_m, e_n) = 1 + \omega_m \omega_{-n} + \omega_m^2 \omega_{-n}^2 + \cdots + \omega_m^{N-1} \omega_{-n}^{N-1} \tag{5.7}$$

$$= 1 + \omega_{m-n} + \omega_{m-n}^2 + \cdots + \omega_{m-n}^{N-1} \,. \tag{5.8}$$

This is an example of a **geometric series,** which you ought to be able to sum on sight: when $m \neq n$ the sum is

$$(e_m, e_n) = \frac{\omega_{m-n}^N - 1}{\omega_{m-n} - 1} \tag{5.9}$$

$$= \frac{\mathrm{e}^{2\pi i(m-n)} - 1}{\mathrm{e}^{2\pi i(m-n)/N} - 1} \,. \tag{5.10}$$

But $\mathrm{e}^{2\pi iK} = 1$ if and only if K is an integer; thus the numerator vanishes, and the denominator does not because we have excluded the case $m = n$. This proves the complex vectors e_n are mutually orthogonal under the inner product (5.1). The case $m = n$ is trivial, since then $\omega_{m-n} = \omega_0 = 1$ and then (5.8) gives at once that

$$(e_n, e_n) = \| e_n \|^2 = N \,. \tag{5.11}$$

We write out what we have in full; (5.3) and (5.4) are as follows

$$z_m = \sum_{n=0}^{N-1} c_n e^{2\pi imn/N} \tag{5.12}$$

$$c_m = \frac{1}{N} \sum_{n=0}^{N-1} z_n e^{-2\pi imn/N} . \tag{5.13}$$

Again the minus sign in (5.13) comes from the complex conjugate in (5.1). The first equation can be interpreted as the construction of a finite length, discretely sampled signal from periodic components with complex amplitudes: it is a finite Fourier synthesis. The second almost symmetrical equation says how to find the expansion coefficients from the original sequence. We can regard the pair as means of approximating the pair of equations (3.10) and (3.11), and they are often used that way; but (5.13) and (5.13) stand on their own as a particular form of Fourier analysis called the **Discrete Fourier Transform**, or **DFT** for short. An important reason why these equations are so useful for approximation is that when N is a power of two (or can be written as the product of a few primes) the numerical evaluation of the DFT can be calculated extremely rapidly. The algorithm is called the **Fast Fourier Transform**, or FFT. We have already met this algorithm in our discussion of convolution.

One of the chief applications of these ideas is to find numerical approximations for an analytic FT, because all too often the necessary integral is hard. Often an asymptotic approximation via the **method of stationary phase,** or the **saddle-point integral** will do the trick (see Bender and Orszag, 1978), and it is a good idea to perform those kinds of calculations anyhow, even if one plans to compute the integral numerically.

Let us define (5.13) as the Discrete Fourier Transform (DFT):

$$\hat{f}_m = \frac{1}{N} \sum_{n=0}^{N-1} f_n \, e^{-2\pi imn/N}, \;\; m = 0, 1, 2, \cdots N-1 \tag{5.14}$$

so that a vector of $\mathbf{f} \in \mathbb{C}^N$ (a complex N-dimensional vector) is mapped into another such vector. The FFT is just a fast way of doing a certain matrix multiply. On the other hand, the Fourier transform is:

$$\hat{f}(\nu) = \int_{-\infty}^{\infty} f(t) \, e^{-2\pi i\nu t} \, dt \tag{5.16}$$

Suppose we are willing to approximate the integral by the **trapezoidal rule**, which is the sum:

$$T[g] = ½h[g(a) + g(b)] + h \sum_{n=1}^{N-1} g(a+nh) \sim \int_a^b g(t) \, dt \tag{5.17}$$

where the spacing $h = (b-a)/N$. Since a and b are at infinity in the original integral (5.16) it should not matter if we keep the factor of one half at the end points in (5.17), since our function should be vanishing small there: we can take a straight sum. In the approximation we can include only finitely many terms, say $2M+1$.

Then we will sample the function $f(t)$ symmetrically about $t = 0$, at the points $t = 0, \pm\Delta t, \pm 2\Delta t, \cdots \pm M\Delta t$; now we have an approximation

$$\tilde{f}(\nu) = \sum_{n=-M}^{M} f(n\Delta t)\, \mathrm{e}^{-2\pi i \nu n \Delta t}\, \Delta t \tag{5.18}$$

Suppose we wish to evaluate the answers at frequencies spaced by $\Delta\nu$, so that evaluation is at $\nu = 0, \Delta\nu, 2\Delta\nu, \cdots 2M\Delta\nu$: then

$$\tilde{f}(m\Delta\nu) = \tilde{f}_m = \sum_{n=-M}^{M} f(n\Delta t)\, \mathrm{e}^{-2\pi i m n \Delta t\, \Delta\nu}\, \Delta t \tag{5.19}$$

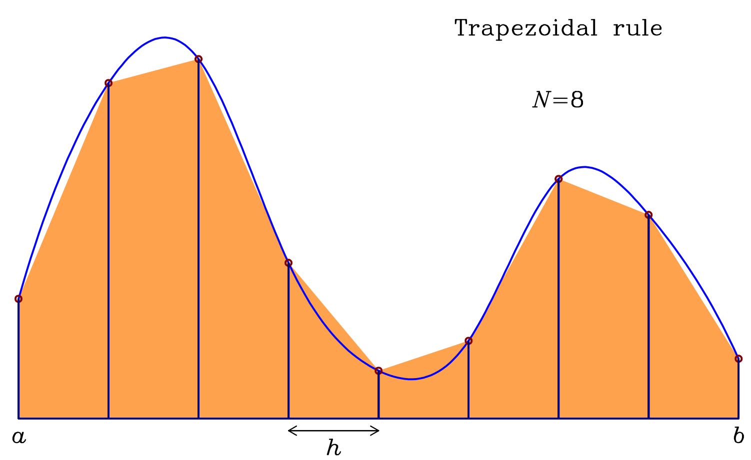
$$= \sum_{k=0}^{2M} f((k-M)\Delta t)\, \mathrm{e}^{-2\pi i m(k-M)\Delta t\, \Delta\nu}\, \Delta t \tag{5.20}$$

$$= \Delta t \mathrm{e}^{-2\pi i m M \Delta t\, \Delta\nu} \sum_{k=0}^{2M} f((k-M)\Delta t)\, \mathrm{e}^{-2\pi i m k \Delta t\, \Delta\nu} \tag{5.21}$$

Comparing this expression with (5.14) we see that the sum is an FFT calculation if we make $N = 2M+1$ and $\Delta t\, \Delta\nu = 1/(2M+1)$. Notice that the input vector is $f_n = f((n-M)\,\Delta t)$ and the output vector is $\tilde{f}_m$ approximates $\hat{f}(m\,\Delta\nu)$.

To get reasonable accuracy out of the trapezoidal rule, we need two things intuitively: (a) A fine enough sampling Δt to capture short wavelength behavior of the original function; (b) Integration over a large enough interval, so that $f(t)$ is effectively zero near the end points, which is equivalent to taking M large enough that $f(M\,\Delta t)$ is very small. These two demands determine an upper bound on Δt and lower bound on M, and therefore choices about $\Delta\nu$, the spacing in frequency at which the results are computed, and the highest frequency, $M\,\Delta\nu$ are constrained. Usually one finds one has to make M much larger than one would like and there are too many points in frequency for convenience.

Figure 5a: Numerical integration with (5.17)



Furthermore, it is easy to show for real vectors f_n in (5.14) that

$$\tilde{f}_{M-m} = \tilde{f}_m^* \tag{5.22}$$

which means that in the DFT approximation (5.21) values for $m > M$ are repeats of those earlier in the vector (with imaginary parts reversed in sign), a property not shared with the true integral $\hat{f}((N-n)\Delta\nu)$. When $m \geq M$ the DFT approximation to $\hat{f}(\nu)$ is worthless.

At a little more sophisticated level, it is well known that for a piece-wise continuous function, the error in the trapezoidal rule falls off like Δt^2. That does not include the error from ignoring the contribution from the integral beyond the ends of the finite interval of integration. For analytic functions (those functions with convergent Taylor series at every point) the error between an infinite sum:

$$T_\infty[g] = h \sum_{n=-\infty}^{\infty} g(nh) \tag{5.23}$$

and the integral of $g(t)$ vanishes much faster than h^2, and usually is of the form $\mathrm{e}^{-c/h}$. So the DFT approximation of an analytic function can be very good, provided a large enough interval is chosen to make the function very small at the end points.

The key to a proper treatment of the error in (5.23) is another important result for Fourier Theory, the **Poisson Sum Formula:** for any sufficiently smooth function, say $f \in \mathcal{S}$

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \hat{f}(m) \tag{5.24}$$

where $\hat{f}$ is the FT of f. To me this is an amazing result, because the sum only samples the function at integer points, while the FT integrates over the whole thing. To use it on (5.23) write $f(n) = g(nh)$ and expand the sum on the right in (5.24)

$$\sum_{n=-\infty}^{\infty} g(nh) = \frac{1}{h}\hat{g}(0) + \frac{1}{h}\sum_{n=1}^{\infty}\hat{g}(m/h) + \frac{1}{h}\sum_{n=-\infty}^{1}\hat{g}(m/h) \tag{5.25}$$

$$= \frac{1}{h}\int_{-\infty}^{\infty} g(t)\, dt + \frac{1}{h}\sum_{m=1}^{\infty}[\hat{g}(m/h) + \hat{g}(-m/h)] \tag{5.26}$$

After multiplying across by h the sum on the left is the trapezium approximation, on the left is the true integral, and the sum is the error in the approximation. This sum can usually be evaluated by an asymptotic method, and when h is small only the first term is important. Remember the Poisson Sum Formula, because it has other applications in time series.

Exercises

1. The FFT is fastest when the number of terms N is a power of two, that is, an even number, while the analysis above yields an odd number for N. How can we ensure an even number instead?

Illustrate your answer by finding a numerical approximation for the FT of the function:

$$f(t) = \frac{1}{\cosh(\pi t)}$$

If you can find the exact answer (either by mathematical skill or the use of works of reference) plot it on your graph of the approximation. Calculate and plot the error of the DFT approximation.

2. Using the code you developed for Question 1, find numerically the FT of the function:

$$g(t) = \exp - |t|^{1/2}$$

Plot the result in an informative way: this means not having everything interesting compressed into one millimeter on the left of the graph. Compute the integral of $\hat{g}(\nu)$ and compare this with $f(0)$; how well do they agree? Is $\hat{g}(\nu)$ positive everywhere?

3. Apply the Poisson Sum Formula to the infinite sum

$$S_1 = \sum_{n=-\infty}^{\infty} \frac{1}{a^2 + n^2} .$$

Use your result to evaluate the sum

$$S_2 = \sum_{n=0}^{\infty} \frac{1}{a^2 + n^2}$$

numerically for $a = 1000$. Give an answer accurate to 6 significant figures. Roughly how many terms of the original series for S_2 would you need to sum to obtain the same accuracy?

6. Multivariate Fourier Transforms

So far we have discussed the Fourier operation on functions of a single variable, which is certainly natural when, as in the last section, the independent variable has been time. When the function in question varies in space, however, we can generalize the notion of the FT. Physically, the model is no longer a sine wave signal like a musical tone, or a electrical current in a wire; now the picture we need is of a plane wave moving over the surface of the ocean, or even a pressure wave moving through the air in 3-dimensional space. For ease of illustration we will mainly work with vectors in 2-dimensional space $\mathbf{x}, \mathbf{k} \in \mathbb{R}^2$, but all the results will remain valid in higher dimensions, unless indicated otherwise.

We define the FT by the integral

$$\hat{f}(\mathbf{k}) = \mathcal{F}[f] = \int_{\mathbb{R}^2} d^2\mathbf{x}\, f(\mathbf{x})\, \mathrm{e}^{-2\pi i \mathbf{k}\cdot\mathbf{x}} \tag{6.1}$$

where $f(\mathbf{x}) = f(x_1, x_2)$ is a complex valued function; this can be stated compactly as $f : \mathbb{R}^2 \to \mathbb{C}$. The integral is performed over the whole x_1, x_2 plane. As before we will not inquire into what kinds of functions are suitable for this operation, but continuously differentiable functions that decay to zero at infinity will certainly be safe. The inverse follows the same pattern as in one dimension:

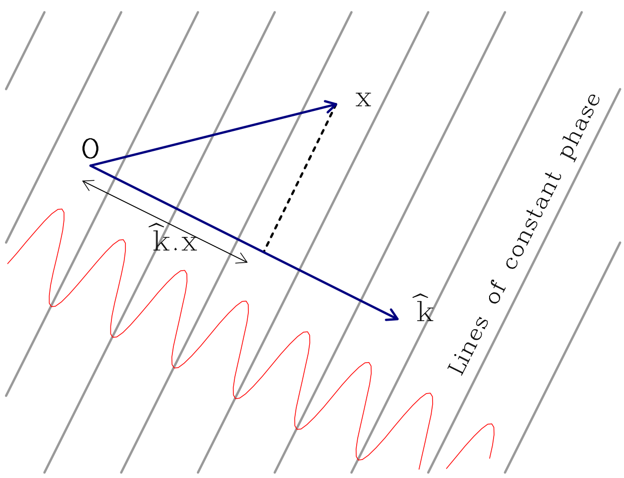
$$f(\mathbf{x}) = \mathcal{F}^{-1}[\hat{f}] = \int_{\mathbb{R}^2} d^2\mathbf{k}\, \hat{f}(\mathbf{k})\, \mathrm{e}^{+2\pi i \mathbf{k}\cdot\mathbf{x}}\,. \tag{6.2}$$

Notice the vector dot product in the exponent. This is the key to understanding the 2-D FFT in physical terms. Equation (6.2) like (1.2) is building up a function from a collection of elementary periodic components, but with the additional complication that each element has a *direction* as well as a wavelength. The complex exponential is

$$\mathrm{e}^{+2\pi i \mathbf{k}\cdot\mathbf{x}} = \cos 2\pi(k_1x_1 + k_2x_2) + i \sin 2\pi(k_1x_1 + k_2x_2)\,. \tag{6.3}$$

The Fourier parameter $\mathbf{k} = (k_1, k_2)$ is the **wavevector** or **wavenumber**; it points in the direction of increasing phase, normal to the wavefront. Imagine plotting the real part of (6.3); you will see a sinusoidal undulation with peaks and troughs running at right angles to the direction $\hat{\mathbf{k}}$ and with a wavelength $\lambda = 1/\sqrt{k_1^2 + k_2^2} = |\mathbf{k}|^{-1}$. The

Figure 6a: A plane sinusoidal wave in 2-D.



argument of the cosine is $2\pi\mathbf{k}\cdot\mathbf{x} = 2\pi\hat{\mathbf{k}}\cdot\mathbf{x}/\lambda$. The integral (6.2) thus represents the summation over a spectrum of waves as a function of their wavenumber, that is, a sum over all wavelengths and all directions: the function $\hat{f}(\mathbf{k})$ gives the (complex) amplitude of the contribution in direction $\hat{\mathbf{k}}$ and wavelength $\lambda = |\mathbf{k}|^{-1}$.

The FT is the means by which a function in the plane can be decomposed into components with differing wavenumber. It is usually easier in a physical problem to understand how a single plane wave interacts with a medium, say by refraction or reflection. Then the behavior of a more complex wave packet can be understood by summing together the responses from the individual planes waves of which it is comprised.

The properties we have encountered for single variable FTs have analogs for the multivariate case. In fact it is usually just possible to substitute the vector argument for the scalar and the result remains valid. You can easily verify that

$$\mathcal{F}[\mathcal{F}[f(\mathbf{x})]] = f(-\mathbf{x}) \tag{6.4}$$

$$\mathcal{F}[f(-\mathbf{x})] = \hat{f}(\mathbf{k})^*, \quad f \text{ real}. \tag{6.5}$$

And The FT still preserves the norm:

$$\int_{\mathbb{R}^2} d^2\mathbf{x}\ |f(\mathbf{x})|^2 = \int_{\mathbb{R}^2} d^2\mathbf{k}\ |\hat{f}(\mathbf{k})|^2 . \tag{6.6}$$

Differentiation introduces the obvious modification to allow for the fact that the gradient is a vector operator:

$$\mathcal{F}[\nabla f] = 2\pi i\mathbf{k}\,\hat{f}(\mathbf{k}) . \tag{6.7}$$

And its converse is:

$$\mathcal{F}[\mathbf{x}\,f(\mathbf{x})] = -\frac{1}{2\pi i}\nabla_{\mathbf{k}}\hat{f} = -\frac{1}{2\pi i}\left(\frac{\partial\hat{f}}{\partial k_1}, \frac{\partial\hat{f}}{\partial k_2}, \frac{\partial\hat{f}}{\partial k_3}\right). \tag{6.8}$$

Convolution is defined in $\mathbb{R}^2$ in the obvious manner:

$$(s*u)(\mathbf{y}) = \int_{\mathbb{R}^2} d^2\mathbf{x}\ s(\mathbf{x})\,u(\mathbf{y}-\mathbf{x}) = \int_{\mathbb{R}^2} d^2\mathbf{x}\ u(\mathbf{x})\,s(\mathbf{y}-\mathbf{x}) \tag{6.9}$$

and then the Convolution Theorem is the same as (4.6):

$$\mathcal{F}[s*u](\mathbf{k}) = \hat{s}(\mathbf{k})\,\hat{u}(\mathbf{k}) . \tag{6.10}$$

Some 2-dimensional FTs can be performed by simply factoring the function into the product of two single variable functions. The most obvious is the Gaussian hump.

$$\mathcal{F}[e^{-\pi\mathbf{x}\cdot\mathbf{x}}] = \int_{\mathbb{R}^2} d^2\mathbf{x}\ e^{-\pi\mathbf{x}\cdot\mathbf{x}}\ e^{-2\pi i\mathbf{k}\cdot\mathbf{x}} \tag{6.11}$$

$$= \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \, e^{-\pi(x_1^2+x_2^2)} \, e^{-2\pi i(k_1 x_1 + k_2 x_2)} \tag{6.12}$$

$$= \int_{-\infty}^{\infty} dx_1 \, e^{-\pi x_1^2} \, e^{-2\pi i k_1 x_1} \times \int_{-\infty}^{\infty} dx_2 \, e^{-\pi x_2^2} \, e^{-2\pi i k_2 x_2} \tag{6.13}$$

$$= e^{-\pi k_1^2} \, e^{-\pi k_2^2} = e^{-\pi \mathbf{k}\cdot\mathbf{k}} \, . \tag{6.14}$$

We see the same self-transform property in two dimensions as in one, and you will easily see that this must be true in all dimensions.

A Gaussian hump has circular symmetry. Many functions of interest in Fourier theory have circular (or, in 3-D, spherical) symmetry. Now we come to a property that has no obvious analog in the 1-dimensional transform. Suppose the function g is circularly symmetric about the origin:

$$g(\mathbf{x}) = G(\,|\,\mathbf{x}\,|\,) \, . \tag{6.15}$$

Is there anything simple we can say about $\hat{g}$? Yes; we discover that $\hat{g}(\mathbf{k})$ is real, and is also circularly symmetric about the wavenumber origin, and can be found from a special kind of transform of the single-argument function G, as follows. We make the change to polar coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ in the FT integral:

$$\mathcal{F}[g] = \int_0^{\infty} r\,dr \int_0^{2\pi} d\theta \, G(r) \, e^{-2\pi i(k_1 r\cos\theta + k_2 r \sin\theta)} \, . \tag{6.16}$$

We indulge in some elementary trigonometry; in the exponent write

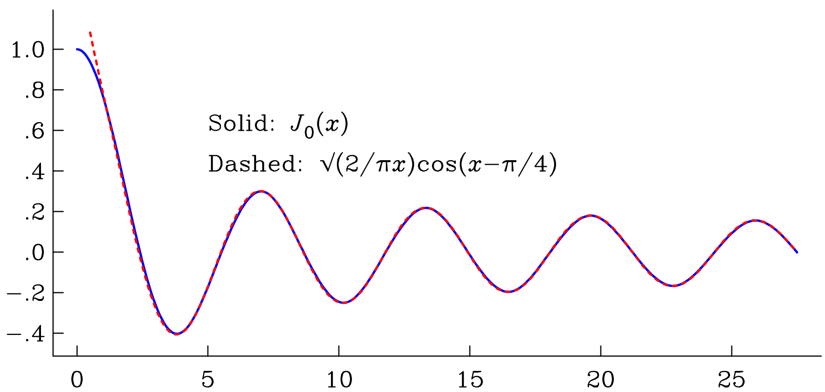
$$k_1 r\cos\theta + k_2 r\sin\theta = kr\cos(\theta - \Phi(\hat{\mathbf{k}})) \tag{6.17}$$

where $k_1 = k\cos\Phi$, $k_2 = k\sin\Phi$ and $k = |\,\mathbf{k}\,|$. Now we can reorder (6.16):

$$\mathcal{F}[g] = \int_0^{\infty} \left[\int_0^{2\pi} d\theta \, e^{-2\pi i k r \cos(\theta - \Phi)} \right] G(r)\, r\, dr \, . \tag{6.18}$$

The value of the integral in brackets is independent of the value of Φ because the θ

Figure 6b: A Bessel function of order zero.



integral is over a complete period of a periodic function, and shifting the argument of the cosine by any constant amount has no effect. (Prove this.) The inner integral becomes

$$\int_0^{2\pi} d\theta \; \mathrm{e}^{-2\pi \mathrm{i} kr\cos\theta} = \int_0^{2\pi} d\theta \, \cos(2\pi kr\cos\theta) = 2\pi \, J_0(2\pi kr) \tag{6.19}$$

where J_0 is called a **Bessel function** (of the first kind and order zero). It is a function that looks a lot like a slowly decaying cosine, (see Figure 6b) and is the simplest member of a big family of special functions obeying a particular kind of ordinary differential equation. Our final result can be written

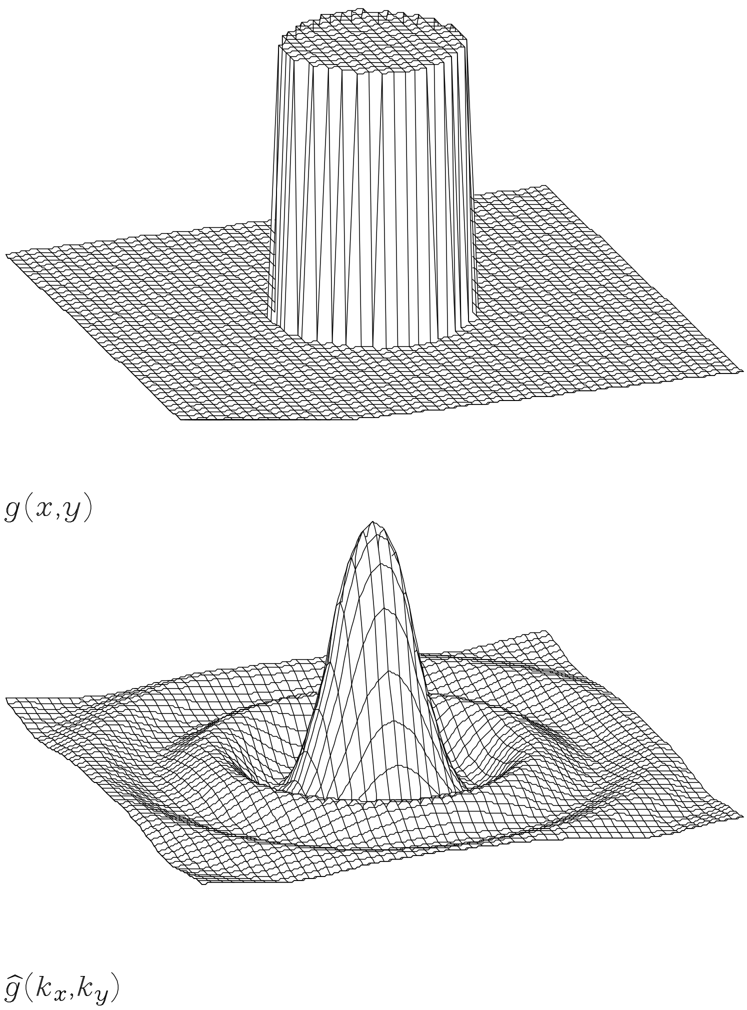
$$\hat{g}(\mathbf{k}) = \mathcal{H}[G](|\mathbf{k}|) \tag{6.20}$$

where

$$\mathcal{H}[G](k) = H(k) = \int_0^{\infty} 2\pi r \, dr \; J_0(2\pi kr)\, G(r)\,. \tag{6.21}$$

The integral in (6.21) is an integral transform called (confusingly) a **Hankel transform**. Notice how the scalar argument function $G(r)$ is mapped into another scalar argument function $H(k)$; even though the Hankel transform arises from

Figure 6c: FT of a circular disk.



2-dimensional problems, it is itself a 1-dimensional transform acting on the radial variable alone. While it is generally true in three and higher dimensions that the centrally symmetric function (a function of $|\mathbf{x}|$ alone, with $\mathbf{x} \in \mathbb{R}^n$) yields an FT that is itself centrally symmetric (a function of $|\mathbf{k}|$), the kind of Hankel transform depends on the value of n, the dimension of the space.

On the previous page we illustrate the 2-D version of the result that the FT of the box is a sinc function. Now we have a circular can-shaped function, and its 2-D FT is a sombrero function, which is of course axisymmetric. Here $H(k) = a\, J_1(2\pi ka)/k$, where $J_1(x) = -\,dJ_0/dx$, another Bessel function, this time of order 1.

Exercises

1. Write the Hankel transform of order zero as

$$\mathcal{H}[g](k) = 2\pi \int_0^\infty f(r)\, J_0(2\pi kr)\, r\, dr$$

Show that the inverse Hankel transform is identical to the forward transform, that is, $\mathcal{H}[g] = \mathcal{H}^{-1}[g]$.

7. Change of Dimension

A scalar function is defined in 3-dimensional space $\mathbb{R}^3$, but observed only on a plane. This situation arises often in potential fields in geophysics, for example, we may know gravity only on the sea surface, even though it is defined above and below that surface. Or in lower dimensions, we may know the magnetic anomaly only on a single track, though it is observable on the sea surface, and in principle defined in all of space. How is the Fourier transform of the function f in $\mathbb{R}^3$, as defined by

$$\hat{f}(\mathbf{k}) = \mathcal{F}_3[f] = \int_{\mathbb{R}^3} d^3\mathbf{x}\, f(\mathbf{x})\, \mathrm{e}^{-2\pi i \mathbf{k}\cdot\mathbf{x}} \tag{7.1}$$

related to the 2-D FT on the plane $z = 0$? The answer is surprisingly easy to prove. Use coordinates (x_1, x_2, x_3) rather than x, y, z. We need a notation that distinguishes 2- and 3-D transform operations and their results; the following is **not** standard, but is quite serviceable. The 2-D FT we need is defined as

$$\hat{f}^0(k_1, k_2) = \mathcal{F}_2[f(x_1, x_2, 0)] \tag{7.2}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_1\, dx_2\, f(x_1, x_2, 0)\, \mathrm{e}^{-2\pi i (k_1 x_1 + k_2 x_2)}\,. \tag{7.3}$$

Now consider the inverse of the full 3-D FT (7.1):

$$f(x_1, x_2, x_3) = \mathcal{F}_3^{-1}[\hat{f}] = \int_{\mathbb{R}^3} d^3\mathbf{k}\, \hat{f}(\mathbf{k})\, \mathrm{e}^{2\pi i \mathbf{k}\cdot\mathbf{x}} \tag{7.4}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_1\, dk_2\, dk_3\, \hat{f}(\mathbf{k})\, \mathrm{e}^{2\pi i (k_1 x_1 + k_2 x_2 + k_3 x_3)}\,. \tag{7.5}$$

If we set $x_3 = 0$ in (7.4) we have

$$f(x_1, x_2, 0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_1\, dk_2\, dk_3\, \hat{f}(\mathbf{k})\, \mathrm{e}^{2\pi i (k_1 x_1 + k_2 x_2)} \tag{7.6}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_1\, dk_2\, \mathrm{e}^{2\pi i (k_1 x_1 + k_2 x_2)} \left(\int_{-\infty}^{\infty} dk_3\, \hat{f}(\mathbf{k}) \right) \tag{7.7}$$

$$= \mathcal{F}_2^{-1}[\int_{-\infty}^{\infty} dk_3 \hat{f}(\mathbf{k})]\,. \tag{7.8}$$

Now all we need do is take the 2-D FT of both sides of (7.8)

$$\hat{f}^0(k_1, k_2) = \int_{-\infty}^{\infty} dk_3 \hat{f}(k_1, k_2, k_3)\,. \tag{7.9}$$

This result shows that the to obtain the 2-D FT on a plane through the origin, you must integrate the full 3-D FT on the line in the wavenumber space that is perpendicular to the plane. As you will easily see, the same argument works in going from the FT over a plane to the FT on a line in the plane. Bracewell calls this the **Slice Theorem**. You don't want to make the mistake of thinking that the 2-D FT on a slice is just a slice through the 3-D FT, but this is an error that is often made.

You might imagine that this is fairly useless, since 3-D FTs are harder to find than 2-D or 1-D transforms. But that is not always true. Here is a simple example. Consider the gravitational potential of a point mass at the origin of coordinates. It is well known that

$$U(\mathbf{x}) = -\frac{Gm}{|\mathbf{x}|} . \tag{7.10}$$

On the plane $z = x_3 = h$ the potential is clearly

$$u(x_1, x_2) = U(x_1, x_2, h) = -\frac{Gm}{\sqrt{x_1^2 + x_2^2 + h^2}} . \tag{7.11}$$

There several reasons why we would like the 2-D FT of u; for example, to perform a convolution over an extended body to calculate its gravity anomaly, when we could use the Convolution Theorem. Equation (7.11) clearly is a function that is circularly symmetric about $(0, 0)$. So according to (6.20), after setting $r^2 = x_1^2 + x_2^2$, we have

$$\mathcal{F}_2[u] = 2\pi \int_0^\infty r\, dr\, \frac{J_0(2\pi r\sqrt{k_1^2 + k_2^2})}{\sqrt{r^2 + h^2}} \tag{7.12}$$

which is not an easy integral. We can discover the answer quite simply in another way.

In place of (9) we write the fundamental differential equation for the gravitational potential, Poisson's equation:

$$\nabla^2 U = -4\pi G \rho(\mathbf{x}) \tag{7.13}$$

where ρ is the density distribution. For a point mass at the origin, this becomes

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_3^2} = -4\pi G m \delta(\mathbf{x}) . \tag{7.14}$$

Here you can imagine that $\delta(\mathbf{x})$ is a very tall, narrow Gaussian function centered on the origin, with unit volume; in the ideal case the function becomes arbitrarily narrow and high, and then represents the idealization of the density distribution of a point mass. We will look at this theory later. For now we need only think about what the 3-D FT is of this function. Without going into details, which you can supply yourself, the FT of the Gaussian is a 3-D Gaussian in $\mathbf{k}$ but scaled so that if w is its width is space, its width in wavenumber must be $1/w$. As w tends towards zero the FT tends to a constant, and because the integral of $\delta(\mathbf{x})$ over all space is unity, that constant must be unity too. Now we take the 3-D FT of (7.13) and we see

$$-4\pi^2 |\mathbf{k}|^2 \hat{U}(\mathbf{k}) = -4\pi G m \tag{7.15}$$

since each derivative on the left causes multiplication by $2\pi i k_j$, and $\mathcal{F}_3[\delta] = 1$.

Therefore, the 3-D FT of the potential due to a point mass at the origin is very simple; it is

$$\hat{U}(\mathbf{k}) = \frac{Gm}{\pi\,|\,\mathbf{k}\,|^2}\,. \tag{7.16}$$

Once again the Fourier transform has made a differential equation into an algebraic equation. This process only works when the boundary conditions are applied at infinity. We want the 2-D FT on the plane $z = h$, not the plane $z = 0$. One way to get this is to shift the point mass to the position $x_3 = -h$; the shift property in three dimensions is

$$\mathcal{F}_3[f(\mathbf{x}+\mathbf{x}_0) = \hat{f}(\mathbf{k})\,\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}_0} \tag{7.17}$$

and so setting $\mathbf{x}_0 = \hat{\mathbf{z}}h = (0, 0, h)$ we find

$$\mathcal{F}_3[U(\mathbf{x}+\hat{\mathbf{z}}h)] = \hat{U}(\mathbf{k})\,\mathrm{e}^{2\pi\mathrm{i}k_3h} \tag{7.18}$$

$$= \frac{Gm}{\pi}\,\frac{\mathrm{e}^{2\pi\mathrm{i}k_3h}}{k_1^2+k_2^2+k_3^2}\,. \tag{7.19}$$

This is the 3-D FT of a point mass at $z = -h$. According to (7.9), all we need do now to find the 2-D FT of u is integrate this equation over k_3:

$$\mathcal{F}_2[u] = \frac{Gm}{\pi}\int_{-\infty}^{\infty} dk_3\,\frac{\mathrm{e}^{2\pi\mathrm{i}k_3h}}{(k_1^2+k_2^2)+k_3^2}\,. \tag{7.20}$$

Since k_1 and k_2 are constants as far as the integral is concerned, and the integrand is an even function of k_3, the integral is nothing more than the FT of f_3 defined by (3.10), and given by (3.11):

$$\hat{u}(k_1, k_2) = \frac{Gm}{\sqrt{k_1^2+k_2^2}}\,\mathrm{e}^{-2\pi h\sqrt{k_1^2+k_2^2}}\,.$$

As advertised, $\hat{u}$ is circularly symmetric in wavenumber.

A Short Table of Multivariate FT Properties

	Name	Property	Comments
1	Scale and shift	$\mathcal{F}[f(a(\mathbf{x}+\mathbf{x}_0))] = \frac{1}{\lvert a \rvert} \hat{f}(\mathbf{k}/a)\, e^{2\pi i \mathbf{k}\cdot\mathbf{x}_0}$	
2	Exponential factor	$\mathcal{F}[f(\mathbf{x})\, e^{2\pi i \mathbf{k}_0\cdot\mathbf{x}}] = \hat{f}(\mathbf{k}-\mathbf{k}_0)$	
3	Double transform	$\mathcal{F}^2[f] = f(-\mathbf{x})$	
4	Reflect in origin	$\mathcal{F}[f(-\mathbf{x})] = \hat{f}(\mathbf{k})^*$	Real f
5	Norm preservation	$\lVert f \rVert = \lVert \hat{f} \rVert$	Complex $L_2(\mathbb{R}^N)$
6	Inner prod preservation	$(f, g) = (\hat{f}, \hat{g})$	Complex $L_2(\mathbb{R}^N)$
7	FT of grad	$\mathcal{F}[\nabla f] = 2\pi i \mathbf{k} \hat{f}$	
8	FT of ∇^2	$\mathcal{F}[\nabla^2 f] = -4\pi^2 \lvert\mathbf{k}\rvert^2 f$	
9	Convolution	$\mathcal{F}[f * g] = \hat{f}\ \hat{g}$	
10	Radial symmetry	$\mathcal{F}[g(\lvert\mathbf{x}\rvert)] = G(\lvert\mathbf{k}\rvert)$	
11	Circular symmetry	$\mathcal{F}[g(\lvert\mathbf{x}\rvert)] = \int_0^\infty g(r) J_0(2\pi \lvert\mathbf{k}\rvert r)\, 2\pi r\, dr$	$\mathbf{x}, \mathbf{k} \in \mathbb{R}^2$
12	Slice Theorem	$\mathcal{F}_2[f(x_1, x_2, 0)] = \int_{-\infty}^{\infty} \hat{f}(k_1, k_2, k_3)\, dk_3$	$\mathbf{x} \in \mathbb{R}^3$