1. A Digression on Dipoles

From the perspective of potential theory, the dipole represents the point source obtained by allowing two point charges of opposite sign to approach each other, in such a way that the product of the charges and the separation $p = qd$ is fixed, as the distance is decreased. This picture is artificial in every realistic system, since there are no point electric dipoles in electricity, there are no point charges in magnetism, and there are no negative sign masses in gravity. None-the-less the dipole is fundamental for static magnetism because it appears as the first term in the SH expansion of an internal magnetic field. When $\mathbf{B} = -\nabla\Psi$, we have in a fully normalized form for fields with interior sources:

$$\Psi(\mathbf{r}) = \sum_{l=1}^{\infty} \left(\frac{1}{r}\right)^{l+1} \sum_{m=-l}^{l} b_{lm} Y_l^m(\hat{\mathbf{r}}) \tag{1}$$

where $r > 0$ here and in all our manipulations. The scalar magnetic potential of an axial dipole (the one symmetric about the z axis) is just the $l = 1$, $m = 0$ term

$$\Psi_D(\mathbf{r}) = \frac{b_{10} Y_1^0(\hat{\mathbf{r}})}{r^2} = \frac{N_{10} b_{10} \cos\theta}{r^2}\,. \tag{2}$$

This agrees of course with the traditional multipole way of writing it: for an axial point dipole at the coordinate origin

$$\Psi_D(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{p\cos\theta}{r^2} \tag{3}$$

where p is the *dipole moment*. The constant μ_0 makes its appearance because the field variable $\mathbf{H}$ is conventionally attached to the scalar potential (recall here $\nabla \times \mathbf{H} = 0$), and the 4π is a feature of the SI system. In that system the units of magnetic dipole moment are $\mathrm{A\,m^2}$, also sometimes written as $\mathrm{J\,T^{-1}}$.

In geomagnetism it is more traditional to use the Gauss coefficients in the semi-normalized expansion; recall that

$$\Psi(\mathbf{r}) = a \sum_{l=1}^{\infty} \left(\frac{a}{r}\right)^{l+1} \sum_{m=0}^{l} \hat{N}_{lm} [g_{lm} \cos m\phi + h_{lm} \sin m\phi]\, P_l^m(\cos\theta)\,. \tag{4}$$

Then for the axial dipole term we extract the single term

$$\Psi_D(\mathbf{r}) = \frac{a^3 g_{10} \cos\theta}{r^2}\,. \tag{5}$$

Notice a simplification here because the normalizing constant $\hat{N}_{10} = 1$. Identifying coefficients in (3) and (5), we see that

$$p = \frac{4\pi a^3 g_{10}}{\mu_0}\,. \tag{6}$$

The multipole interpretation of (3) leads to another way of expressing Ψ_D. Since the separation of the fictitious charges is a vector quantity, so is the dipole moment. In (2)-(5) we have looked only at an axial dipole, that is, the one with circular symmetry about $\hat{\mathbf{z}}$. Clearly (3) can be written in a way that works for a dipole pointing in any direction:

$$\Psi_D(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \tag{7}$$

where $\mathbf{p}$ is the vector dipole moment. You will ready confirm that the Gauss coefficients g_{11}, h_{11} or the fully normalized $b_{1,\pm 1}$ express the potentials due to x and y components of the vector $\mathbf{p}$. For the general case (6) must be revised:

$$p = \frac{4\pi a^3 (g_{10}^2 + h_{10}^2 + g_{11}^2)^{½}}{\mu_0} . \tag{8}$$

Another way that (7) has already been expressed, in the Equivalent Source Theorem, is

$$\Psi_D(\mathbf{r}) = -\frac{\mu_0}{4\pi} \mathbf{p} \cdot \nabla \frac{1}{r} . \tag{9}$$

In electromagnetic theory the physical model for a magnetic dipole is not the pair of opposite sign poles, but the tiny current loop. To the accuracy set by classical physics, electrons and other elementary particles are point dipoles; and the circulation of electrons within atoms comprises elementary current loops. We will merely state a number of results about dipoles related to this approach. Proofs will be found in most books on classical electromagnetism or in *Foundations of Geomagnetism*, Chapter 2. The natural vehicle for this theory is of course the vector potential $\mathbf{A}$ with $\mathbf{B} = \nabla \times \mathbf{A}$. Then the analog of (9) is

$$\mathbf{A}_D(\mathbf{r}) = -\frac{\mu_0}{4\pi} \mathbf{p} \times \nabla \frac{1}{r} \tag{10}$$

where $\mathbf{p}$, the dipole moment can be computed for a set of divergence-free currents by

$$\mathbf{p} = \tfrac{1}{2} \int_V (\mathbf{s} \times \mathbf{j}) \, d^3\mathbf{s} . \tag{11}$$

We will demonstrate the equivalence of (9) and (10) later. For an elementary circular current loop with radius r and normal $\hat{\mathbf{n}}$, (11) integrates to

$$\mathbf{p} = \hat{\mathbf{n}} \pi r^2 j \tag{12}$$

so that the magnitude of the dipole moment is just the area times the current; it is this picture that motivates the usual units of amp meter-squared for dipole moment.

What happens if we carry through the calculation to produce the actual field, rather than merely displaying the potentials? The grand result, proved in the next Section is this: the vector field $\mathbf{B}$ at $\mathbf{r}$ due to a dipole $\mathbf{p}$ at O is

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\nabla\left(\mathbf{p}\cdot\nabla\frac{1}{r}\right) = -\frac{\mu_0}{4\pi}\mathbf{p}\cdot\nabla\nabla\frac{1}{r} \tag{13}$$

$$= \frac{\mu_0}{4\pi}\left(\frac{3\mathbf{p}\cdot\mathbf{r}\,\mathbf{r}}{r^5} - \frac{\mathbf{p}}{r^3}\right). \tag{14}$$

Finally, we can use (13) to show the equivalence of (9) and (10). Note that since **p** is constant, vector identity #5 for the curl shows that

$$\mathbf{A}_D(\mathbf{r}) = -\frac{\mu_0}{4\pi}\boldsymbol{\nabla}\times\left(\frac{\mathbf{p}}{r}\right). \tag{15}$$

Then, taking the curl of (15) together with identity #9 yields

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\left(\frac{\mathbf{p}}{r}\right) \tag{16}$$

$$= -\frac{\mu_0}{4\pi}\left[\nabla\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{p}}{r}\right) - \nabla^2\left(\frac{\mathbf{p}}{r}\right)\right]. \tag{17}$$

Again, since **p** is constant, the second term is just $\mathbf{p}\,\nabla^2 1/r$ which vanishes, and we can use identity #4 to expand the remaining term:

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\nabla\left(\mathbf{p}\cdot\nabla\frac{1}{r} + \frac{1}{r}\boldsymbol{\nabla}\cdot\mathbf{p}\right) \tag{18}$$

$$= -\frac{\mu_0}{4\pi}\nabla\mathbf{p}\cdot\nabla\frac{1}{r} = -\frac{\mu_0}{4\pi}\mathbf{p}\cdot\nabla\nabla\frac{1}{r} \tag{19}$$

which is identical to (13).

2. Fourier Theory for Magnetic Stripes

We make a series of assumptions concerning the source magnetization for the seafloor-spreading stripes universally observed on ocean ridges, and in most other places in the ocean basins. These assumptions are essential if the source magnetization is to be modeled, but should not be taken too seriously as representing the state of the crust. First we assume the magnetization is invariant in the y direction: this is often called the 2-dimensional approximation, and is obviously just that whenever a proper survey is available instead of a single profile. The intensity of magnetization will vary only with x, and the magnetic layer is thin compared to the water depth; sometimes it will be taken to be a thin sheet of dipoles for simplicity, sometimes a definite thickness is assumed. We will study the simplest model, of course. The direction of magnetization is assumed to be constant *Mo*. That direction can be assumed to be known, or we may want to treat it as an unknown parameter of the model. The seafloor is treated as flat in our simplified exposition, but these days that is not usually assumed. Instead the Fourier theory given for the Bouguer terrain correction in the Gravity Notes is adapted. With all this baggage on board we can proceed.

The ship track runs in the x direction on the surface $z = h$, and the layer is concentrated on the plane $z = 0$, with $z > 0$ upwards (not the normal geomagnetic convention). The plan is to find the Fourier transform (FT) of the total field anomaly at the sea surface due to an infinite line source of dipoles at a depth z. The anomaly from an extended sheet is the convolution of a set of such sources with the magnetization intensity function.

We begin with a line source at the coordinate origin and an observer at $(x, 0, z)$ with $z > 0$. We need to integrate the source dipoles in the y direction. Consider ψ the scalar potential of an infinite line of dipoles along the y axis as seen at $(x, 0, z)$:

$$\psi = \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \, dy \tag{20}$$

$$= \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} \frac{p_x x + p_y y + p_z z}{(x^2 + y^2 + z^2)^{3/2}} \, dy \, . \tag{21}$$

The integral over the middle term vanishes, because the integrand is an odd function. So now

$$\psi = \frac{\mu_0}{4\pi} (p_x x + p_z z) \int_{-\infty}^{\infty} \frac{dy}{(x^2 + z^2 + y^2)^{3/2}} \tag{22}$$

$$= \frac{\mu_0}{4\pi} \frac{p_x x + p_z z}{x^2 + z^2} \int_{-\infty}^{\infty} \frac{dt}{(1 + t^2)^{3/2}} \tag{23}$$

$$= \frac{\mu_0}{2\pi} \frac{p_x x + p_z z}{x^2 + z^2} \tag{24}$$

since the integral in t is just two.

Consider the potential as x varies, but z is held constant. The x Fourier transform of the potential is best done next. Recall the following results from Fourier theory:

$$\mathcal{F}\left[\frac{z}{x^2 + z^2} \right] = \pi \, \mathrm{e}^{-2\pi z |k|} \quad \text{and} \quad \mathcal{F}\,[x\, f(x)] = -\frac{1}{2\pi i} \, \partial_k \hat{f} \, . \tag{25}$$

With these in hand we have

$$\hat{\psi}(k) = \int_{-\infty}^{\infty} \psi(x, z) \, \mathrm{e}^{-2\pi i k x} \, dx \tag{26}$$

$$= \frac{\mu_0}{2\pi} \left[-\frac{p_x}{2\pi i} \, \partial_k + z p_z \right] \pi \mathrm{e}^{-2\pi z |k|} \tag{27}$$

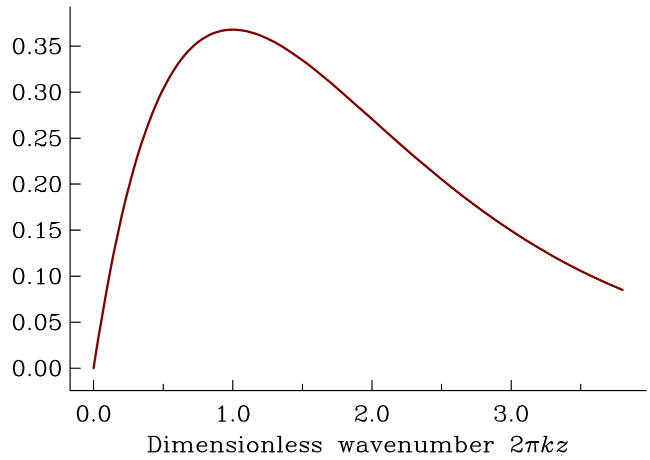

Figure 5.4.1.1: Amplitude factor in the Earth filter

$$= \tfrac{1}{2}\,\mu_0 z[-\mathrm{i}p_x\, sgn(k) + p_z]\mathrm{e}^{-2\pi z|k|}\,. \tag{28}$$

Next we need the FT of the field components associated with the line of dipoles. Recall from the Gravity Notes that for a harmonic function, like ψ,

$$\mathcal{F}\,[\partial_x\psi] = 2\pi\mathrm{i}k\,\hat{\psi}(k) \quad \text{and} \quad \mathcal{F}\,[\partial_z\psi] = -2\pi\,|k|\,\hat{\psi}(k)\,. \tag{29}$$

The total field anomaly of the line source is given by

$$\Delta T_{LS}(x) = \hat{\mathbf{B}}_0\cdot\Delta B = -\hat{\mathbf{B}}_0\cdot\nabla\psi = -\beta_x\partial_x\psi - \beta_z\partial_z\psi \tag{30}$$

where $\mathbf{B}_0 = (\beta_x, \beta_x, \beta_z)$, the unit vector along the local main field. The y component of the gradient vanishes in this 2-D geometry, so there is no contribution from β_y. Combining (28), (29), and (30) we find the FT of the total field anomaly from a buried line source

$$\Delta\hat{T}_{LS}(k) = -\mathcal{F}\,[\beta_x\partial_x\psi + \beta_z\partial_z\psi] \tag{31}$$

$$= -2\pi\mathrm{i}k\beta_x\hat{\psi}(k) + 2\pi\,|k|\,\beta_z\hat{\psi}(k) \tag{32}$$

$$= \tfrac{1}{2}\,\mu_0 z\,[-2\pi\mathrm{i}k\beta_x + 2\pi\,|k|\,\beta_z]\,[-\mathrm{i}p_x\, sgn(k) + p_z]\mathrm{e}^{-2\pi z|k|} \tag{33}$$

$$= \tfrac{1}{2}\,\mu_0[\mathrm{i}\beta_x sgn(k) - \beta_z]\,[\mathrm{i}p_x\, sgn(k) - p_z]\,2\pi\,|k|\,z\,\mathrm{e}^{-2\pi z|k|}\,. \tag{34}$$

The final step is to generate an anomaly by combining together the effects from a set of line sources arranged as a horizontal sheet. Suppose the strength of the dipoles varies as $m(x)$; then the aggregate magnetic anomaly is given by the integral

$$\Delta T(x) = \int_{-\infty}^{\infty} m(y)\,\Delta T_{LS}(x-y)\,dy = m * \Delta T_{LS} \tag{35}$$

where $*$ denotes convolution. Thus the Convolution Theorem gives us the

FT of the complete anomaly:

$$\Delta\hat{T}(k) = \tfrac{1}{2}\,\mu_0 \hat{m}(k)\,[\mathrm{i}\beta_x sgn(k) - \beta_z]\,[\mathrm{i}p_x\, sgn(k) - p_z]\; 2\pi\,|\,k\,|\,z\, \mathrm{e}^{-2\pi z|k|} \qquad (36)$$

$$= \tfrac{1}{2}\,\mu_0\, \hat{m}(k)\,\Theta(k)\, 2\pi\,|\,k\,|\,z\, \mathrm{e}^{-2\pi z|k|} \qquad (37)$$

where $\Theta(k)$ is the product of the two bracketed factors. This result can be interpreted as a wavenumber filter of the original intensity signal $m(x)$. The name *Earth filter* has been suggested. The amplitude is controlled by the factor $2\pi\,|\,k\,|\,z \exp(-2\pi\,|\,k\,|\,z)$. As shown in the figure, the filter attenuates the high wavenumber, or short wavelength components of m. It is simply the effect of upward continuation from the seabed to the sea surface, which removes the high-frequency variations. But the filter also removes the long wavelength energy too, through the factor k. As we might expect a constant magnetization has no magnetic anomaly at all. So the filter is a weak band-pass filter with its peak at $k = 1/2\pi z$. In water 2 km deep this corresponds to a wavelength of about 12.5 km.

The factor $\Theta(k)$ in (37) introduces phase shifts that distort the shape of the anomaly. The two factors comprising Θ are both of exactly the same functional form: the first is due to the inclination of the ambient main field, which is known, the second from the inclination of the magnetization vector, which may not be known if the anomalies are on old crust. If the magnetization and the local main field were both vertical (a mid ocean ridge at the North magnetic pole) then there would be no phase shifts. In fact the function Θ is controlled by only one parameter, not two as it would appear. Suppose we write

$$\beta_z = \beta \cos A_\beta, \quad \beta_x = \beta \sin A_\beta \qquad (38)$$

$$p_z = p \cos A_p, \quad p_x = p \sin A_p\,. \qquad (39)$$

Then A_β is the angle between vertical and the projection of the vector $\hat{\mathbf{B}}_0$ in the x-z plane, and similarly for A_p. Then one can verify that

$$\Theta(k) = \beta\, \mathrm{e}^{-\mathrm{i}A_\beta sgn(k)}\; p\, \mathrm{e}^{-\mathrm{i}A_p sgn(k)} \qquad (40)$$

$$= \beta p\, \mathrm{e}^{-\mathrm{i}(A_\beta + A_p)\, sgn(k)}\,. \qquad (41)$$

This equation shows that $\Theta(k)$ does not affect the magnitude of the signal, since clearly $|\,\Theta(k)\,|$ = constant, independent of k. And phase shift depends only on the sum of the two angles. The idea of *reduction to the pole* is to choose an angle $A = A_\beta + A_p$ so that filtering the observed anomaly signal with $\Theta(k)$ produces a symmetric-looking anomaly. Here we must assume that the magnetization pattern is of constant magnitude, with reversals of sign, corresponding to the reversal pattern of the main field.

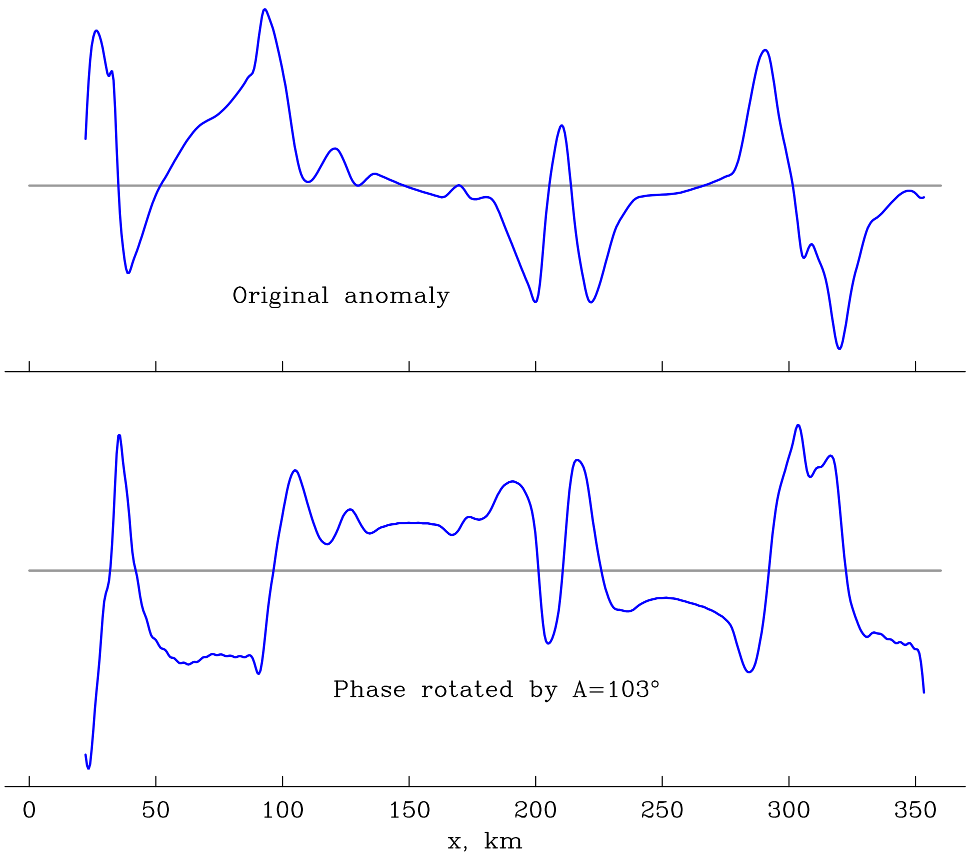

Figure 5.4.1.2: Measured anomaly, and the same anomaly reduced to the pole.

We can illustrate these ideas with an example. In the figure we see a magnetic anomaly recorded in the Indian Ocean near the equator so that A_β is quite large; see McKenzie and Sclater, Geophys. J. R. astr. soc., v 24, 437-528, 1971. The anomaly pattern is quite different from the symmetrical forms seen near spreading centers at high latitudes, and these turn out to be quite old anomalies. By trial and error I found a value of $A = -103°$ applied with $\Theta(k)$ yields a much more familiar blocky anomaly as shown in the lower graph.

For much more on this topic consult Blakely, *Potential Theory in Gravity and magnetic Applications.*