

## 6. ELECTROMAGNETIC INDUCTION IN EARTH'S CRUST AND MANTLE

In the late 19th century Schuster discovered that a minor part of the diurnal variation field originated within the earth: this part of the variation is due to eddy currents induced within the conducting earth by the larger external field. The response of Earth to time variations in the externally generated geomagnetic field can be interpreted in terms of electrical conductivity variations with depth. The estimation of geomagnetic response functions and their interpretation in terms of mantle electrical conductivity structure dates from the end of the last century.

There are two different approaches: *magnetotelluric sounding* and *geomagnetic deep sounding*. The first uses relatively high frequency variations (sometimes as high as 1 kHz) and requires the measurement of electric as well as magnetic fields; it is suitable mainly for shallow structures, in the upper crust or above. As the name suggests the magnetic sounding employs only magnetic fields usually measured by arrays of instruments, for probing the mantle.

### 6:1 Skin Depth

We shall ignore magnetic permeability and electric polarizations, fairly good approximations in the deep crust and mantle. Then the equations we must work with are the same as those we studied in 4.3, derived from pre-Maxwell's equations in a stationary conductor; we get the vector diffusion equation for  $\mathbf{B}$ . There are three differences in the approach here. First, we cannot so readily ignore the boundary conditions, which in our discussion of the core had no major effect on the physics. The interface of the conductor with the insulating atmosphere is critical because it is there that we make our measurements. The second, perhaps slightly related, question concerns the temporal behavior of the magnetic field, which we now think of as typically periodic, not exponentially decaying with time. Finally, we cannot treat the conductor as uniform for any but the most superficial analysis because in the mantle alone  $\sigma$  increases by at least four orders of magnitude.

Despite the last remark, we will perform a simple study of a uniform conductor. In our present problem energy is being supplied by a fluctuating external field, created above the atmosphere by the solar wind and its interplay with the magnetosphere. So we write (118), the vector diffusion equation

$$\nabla^2 \mathbf{B} = \mu_0 \sigma \partial_t \mathbf{B}. \quad (172)$$

Later we will return to spatially variable  $\sigma$ , but to get started we will make  $\sigma$  constant. We erect a traditional Cartesian coordinate system with  $z$  positive downward. Then above ground,  $z < 0$ , the atmosphere is an insulator; below in  $z > 0$ , is a uniformly conducting halfspace with conductivity  $\sigma$ . The wavelengths of the external fields are very long (the ring current is typically at 10 Earth-radii) and so for a local problem we may consider a uniform magnetic field in the atmosphere, in the  $x$  direction, varying in time as  $e^{i\omega t}$ :

$$\mathbf{B} = \hat{\mathbf{x}} B_0 e^{i\omega t}, \quad z \leq 0. \quad (173)$$

Below ground, to match continuity at  $z = 0$ , we have a horizontal field too, but it can vary in the  $z$  direction:

$$\mathbf{B} = \hat{\mathbf{x}} b(z) e^{i\omega t}, \quad z \geq 0. \quad (174)$$

Plugging (174) into (172) gives

$$i\omega\mu_0\sigma\hat{\mathbf{x}}b(z)e^{i\omega t} = \hat{\mathbf{x}}e^{i\omega t}\frac{d^2b}{dz^2} \quad (175)$$

or

$$\frac{d^2b}{dz^2} = i\omega\mu_0\sigma b(z). \quad (176)$$

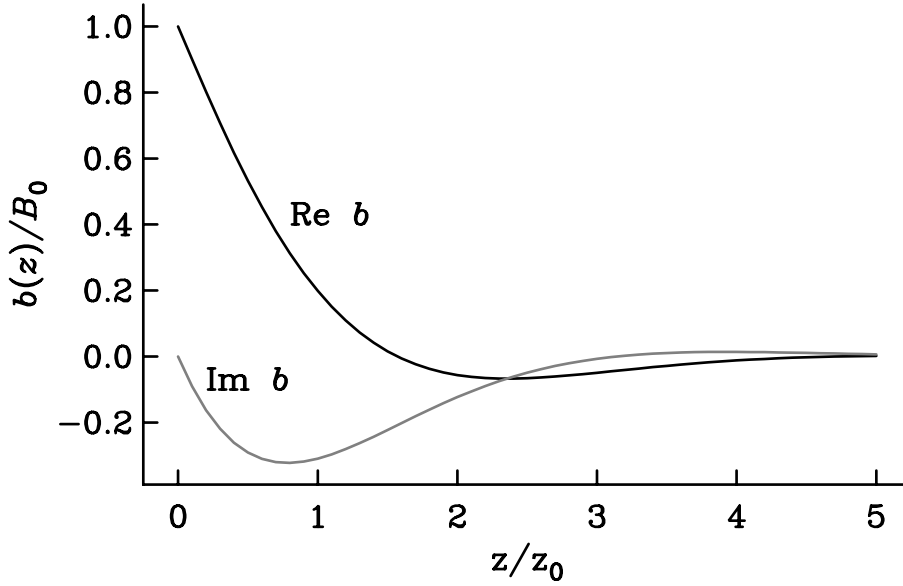


Figure 6.1.1

This is the simplest second-order ordinary differential equation (it is just  $y'' = k^2y$  dressed up), and its general solution is

$$b(z) = Ae^{(1+i)z/z_0} + Be^{-(1+i)z/z_0} \quad (177)$$

where

$$z_0 = \sqrt{2/\omega\mu_0\sigma} = 1/\sqrt{\pi\mu_0 f\sigma}. \quad (178)$$

The first term in (177) is a growing (and oscillating) exponential function, a term that increases with depth: this term is analogous to the magnetic fields of internal origin we saw in the ordinary SH expansion. Since the source of the energy for the system is above ground, we conclude that  $A = 0$ . The remaining term dies away exponentially with depth, and the characteristic e-folding scale is  $z_0$  which is the *skin depth*. Matching with the atmospheric field at  $z = 0$  gives us

$$\mathbf{B}(z) = \hat{\mathbf{x}}B_0e^{-z/z_0}[\cos z/z_0 - i\sin z/z_0]. \quad (179)$$

It is easily seen from (178) that the high-frequency variations correspond to very shallow disturbances below ground, while the long period fields penetrate more deeply. You will easily verify that both the electric field and electric currents (both horizontal, in the  $y$  direction) fall off at the same rate as the magnetic field.

**Table of Skin Depths**

material	$\sigma$ , S/m	1 year	1 month	1 day	1 hour	1 sec	1 ms
core	$3 \times 10^5$	4 km	770 m	200 m	40 m	71 cm	23 mm
L. mantle	10	900 km	170 km	46 km	9 km	160 m	5 m
seawater	3.4	1500 km	300 km	80 km	16 km	270 m	9 m
sediments	0.1	9000 km	1700 km	460 km	95 km	1.6 km	50 m
U. mantle	0.001	$10^4$ km	$10^4$ km	4600 km	950 km	16 km	500 m
igneous	$1 \times 10^{-5}$	$10^5$ km	$10^5$ km	$10^4$ km	9500 km	160 km	5 km

The table shows skin depths for a variety of typical Earth environments and a large range of frequencies. The first line shows that for practical sounding the core is effectively a perfect conductor into which the external magnetic field does not penetrate. The first column shows that 1-year variations (which are not the lowest frequency external fields – there is an 11-year cycle associated with solar sunspot activity) can penetrate into the lower mantle and even to the bottom of the mantle. The very large range of conductivities found in the crust points to the need for a corresponding large range of frequencies in electromagnetic sounding of that region.

## 6:2 Variable $\sigma$ – Magnetotelluric Sounding

As we already noted, the magnetotelluric (MT) method relies on simultaneous measurements of time series of the electric and magnetic fields at the surface. The traditional development, which we will follow, works with the electric field  $\mathbf{E}$  rather than with  $\mathbf{B}$ . As in the previous section we use a flat-Earth approximation and a uniform exciting field above the ground; the only difference is that now the conductivity is allowed to vary with  $z$ . The idealized experimental arrangement to measure two horizontal components,  $B_x$  and  $E_y$  at a single site. We will see in a moment that if the ground is truly vertically stratified the electric and magnetic fields will be mutually perpendicular. Then think of the magnetic field as causing eddy currents in the ground, currents that are evident from the electric fields seen at the surface. The governing equations are linear, and so a periodic magnetic field with frequency  $f$  excites currents and a responding electric field at the same frequency. In the real world the fluctuations of magnetosphere are far from sinusoidal, but that difficulty is overcome by Fourier transforming the actual magnetic and electric time series, thereby isolating a particular frequency by extracting a particular Fourier component. Fourier decomposition is done at a series of frequencies motivated by the skin-depth analysis: the long-period variations penetrate deep into the ground and give some kind of average measure of the deep conductivity, while the short-period energy probes the near-surface structure.

Since the amplitude of the source field is of no interest to us in this problem, our objective must be to find a measure

that removes dependency on properties of the source. In magnetotelluric sounding one divides the electric field complex Fourier amplitude at a given frequency, by the complex amplitude of the magnetic field. The resulting complex number carries the electromagnetic response of the ground; it is complex because the currents induced into the ground lag behind the forcing magnetic field, and the phase lag is contained in the complex response, as well as the amplitude. Let us analyze the simplest physical system.

We assume all the fields vary periodically in time as  $e^{i\omega t}$ ; then we write two of the pre-Maxwell equations and Ohm's law

$$\nabla \times \mathbf{E} = -i\omega \mathbf{B} \quad (180)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad (181)$$

$$\mathbf{J} = \sigma \mathbf{E}. \quad (182)$$

We take the curl of (180), then substitute from (181) and (182):

$$\nabla \times \nabla \times \mathbf{E} = -i\omega \mu_0 \sigma \mathbf{E}. \quad (183)$$

In the geometry of the previous section nothing varies in the  $x$  or  $y$  directions, and that remains true here also. We look for solutions in this form, with variations in the  $z$  direction only. Again we set the  $\mathbf{B}$  field in the  $\hat{\mathbf{x}}$  direction. Then it follows from (181) that the vectors  $\mathbf{J}$  and hence  $\mathbf{E}$  (through Ohm's law) have no  $x$  or  $z$  components; they are vectors in the  $y$  direction. Thus  $\nabla \cdot \mathbf{E} = \partial_y E_y$ ; but there is no gradient in  $y$  and so we must have that  $\nabla \cdot \mathbf{E} = 0$ . Hence the familiar vector identity shows that  $\nabla \times \nabla \times \mathbf{E} = -\nabla^2 \mathbf{E}$ ; from this, bearing in mind there are no  $x$  or  $y$  gradients, we conclude that (183) simplifies to

$$\partial_z^2 E_y = i\omega \mu_0 \sigma E_y \quad (185)$$

which is an ordinary differential equation governing  $E_y$ . We will concentrate on solving this equation. To find  $B_x$  we use (180)

$$B_x = \frac{1}{i\omega} \partial_z E_y. \quad (186)$$

In the previous section we obtained one boundary condition from the fact that of two linearly independent solutions, one grows with depth. That works here too, but particularly for numerical solutions, a boundary condition applied at  $z = \infty$  is awkward. It is easier to work in a system with finite  $z$  extent; then a natural boundary condition is to set  $E_y = 0$  at the base, which we will put at  $z = h$ . Physically this is equivalent to placing a perfect conductor there; a small amount of energy does flow upward now – it is the energy reflected back from the perfect conductor. Obviously if  $h$  is many skin depths (at the lowest frequency), solutions to the finite problem will be indistinguishable from those in a halfspace.

To get the second boundary condition recall that the measurements we will be using are normalized by the source field. In fact a common type of measurement is the magnetotelluric *admittance function*, defined by

$$c = -\frac{E_y(0)}{i\omega B_x(0)} = -\frac{E_y(0)}{E'_y(0)} \quad (187)$$

where the prime means derivative on  $z$ . Recall that  $E_y(0)$  and  $B_x(0)$  are complex functions of frequency, derived from the field time series by Fourier analysis. Thus (187) shows that  $c$  is a complex function of frequency available from observation, and it is also computable from a model. We can solve (185) with a conductivity profile  $\sigma$  choosing an arbitrary nonzero value for  $E_y$  at  $z = 0$ ; whatever value is used for the top boundary condition, disappears in the normalization performed in (187). Alternatively, and more precisely,  $c = E_y(0)$  is found by solving (185) with the boundary conditions  $E_y'(0) = -1$  and  $E_y(h) = 0$ .

This prescription allows us to compare observed values of  $c$  with predictions from a model  $\sigma$ , and then of course we can attempt to adjust the model until its predictions match the measurements: this is another example of an inverse problem. After the magnetic annihilator and the equivalent source theorem we have become used to the situation where having good agreement between a model and data does not guarantee any degree of approximation to the true structure in the Earth. However, in this case there is a uniqueness theorem – a function  $\sigma(z)$  that achieves exact matching of  $c$  for all  $\omega$  is the only possible solution.

There is an alternative to the admittance which provides some insight into the Earth's response. We ask, How does  $c$  behave over a uniform halfspace? You will readily verify that, when  $\sigma = \sigma_0 = \text{constant}$ ,

$$c(\omega) = \frac{1 - i}{\sqrt{2\omega\mu_0\sigma_0}} = \frac{1}{2}z_0(1 - i) \quad (188)$$

where you may recall  $z_0$  is the skin depth in (178). Thus the magnitude of the admittance  $|c(\omega)|$  is  $1/\sqrt{2} = 0.707$  times the skin depth. Approximating the system as though it were a uniform halfspace, we can calculate its effective resistivity directly from (188), a quantity called the *apparent resistivity*:

$$\rho_a = \omega\mu_0|c(\omega)|^2. \quad (189)$$

By plotting  $\sqrt{2}|c|$  on the depth axis and  $\rho_a$  as a resistivity, we can obtain a very crude picture of the electrical structure of the ground directly from admittance data.

As a method of model construction the use of apparent resistivity is obviously too primitive. The MT problem in one dimension is probably the best-understood nonlinear inverse problem in geophysics. We could devote a whole quarter to exploring the various approaches. For one powerful technique see Chapter 5 of my book, *Geophysical Inverse Theory*.

*Exercise:*

Solve (185) for a non-constant function  $\sigma$  of your choice. Plot  $c$  and  $\rho_a$  as functions of frequency and then plot  $\rho_a$  against  $\sqrt{2}|c|$ ; also plot  $1/\sigma(z)$  on the second plot.

### 6:3 Global Electromagnetic Sounding - Results

In the huge frequency range of  $3 \times 10^{-8}$  to 0.001 Hz, corresponding to periods of one year to 15 minutes, the fluctuating geomagnetic field at the Earth's surface is driven by currents circulating within the magnetosphere, a giant loop called the

*ring current*. The main constituent is a small westward drift of protons whose velocity direction at any instant is mainly north or south as the particles bounce between mirror points at high latitudes. To an observer on the Earth's surface the field appears to be almost uniform, or a  $Y_1^0$  spherical harmonic in the external potential (with the  $z$  coordinate aligned with the main dipole). When a radially stratified conductor is subjected to a varying uniform field, the toroidal currents that flow have a  $Y_1^0$  geometry, and the internally generated response is  $Y_1^0$  also. For a radially stratified conductor, an externally generated field with  $Y_l^m$  geometry produces an internal part with the same symmetry. The ratio of the internal to external parts of the SH expansion at a particular frequency can be used as an electromagnetic response measure as we will now sketch.

Considering only axisymmetric fields, those with  $m = 0$ , we write the SH expansion:

$$\Psi(r, \theta) = a \sum_l \left[ a_l \left( \frac{r}{a} \right)^l + b_l \left( \frac{a}{r} \right)^{l+1} \right] P_l(\cos\theta) \quad (190)$$

where  $a_l$  and  $b_l$  are the external and internal Gauss coefficients for some frequency  $\omega$ . As we have seen it is possible to obtain  $a_l, b_l$  by Fourier analysis of surface observatory time series. The complex ratio  $Q_l = b_l/a_l$  is a response measure sensitivity to the Earth's conductivity. Weidelt (*Zeitschrift für Geophysik*, 38, pp 257-89, 1972) showed that  $Q_l$  could be mapped into a function that is the 1-dimensional admittance of an equivalent flat-Earth conductor with

$$c = \frac{a}{l(l+1)} \frac{l - (l+1)Q_l}{1 + Q_l}. \quad (191)$$

This response measure has become popular. Constable (*J. Geomag, Geoelec.*, 45, pp 707-28, 1993) has performed an analysis of world-wide data in this form. Figure 6.3.1 shows the global admittance over the period range 3 days to half a year. The next figure, Figure 6.3.2, shows a range of models compatible with these admittances. Here is the caption for this figure, quoting from Constable's paper: *Curve 1 has the minimum 2-norm of the first derivative of log  $\sigma$  vs log depth; curve 2 the minimum-norm of second derivative. Curve 3 is the same as 1 but with the penalty on the derivative relaxed at 660 km, and curve 4 is the same as 3 except with a prejudice for  $4 \times 10^{-3}$  S/m applied between 200 and 410 km. All models terminate with an infinite conductor at 3,000 km. measurements of lower mantle materials by Shankland: pw = perovskite plus wustite, p=perovskite, both with 11% iron.*

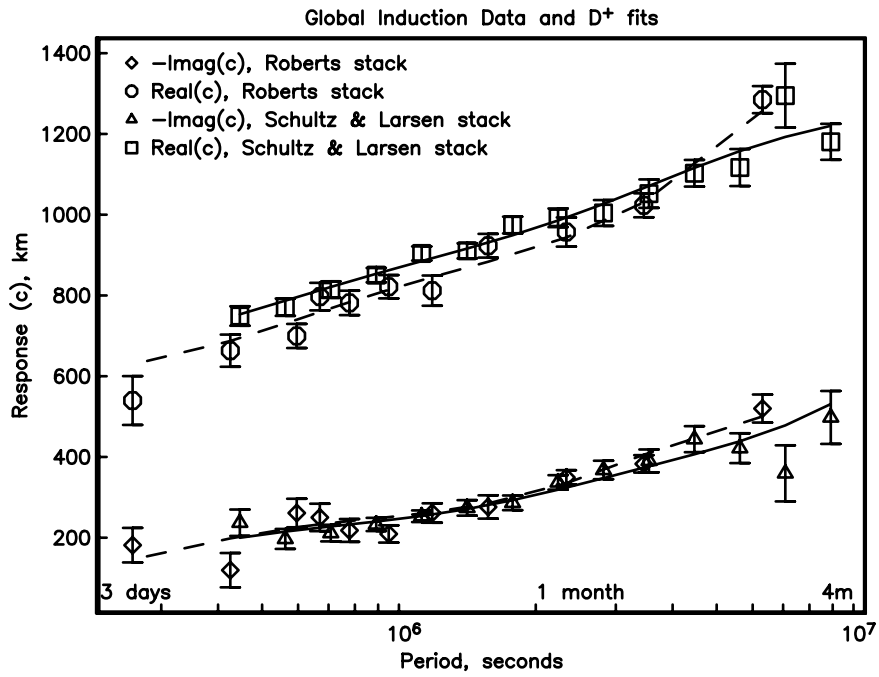


Figure 6.3.1

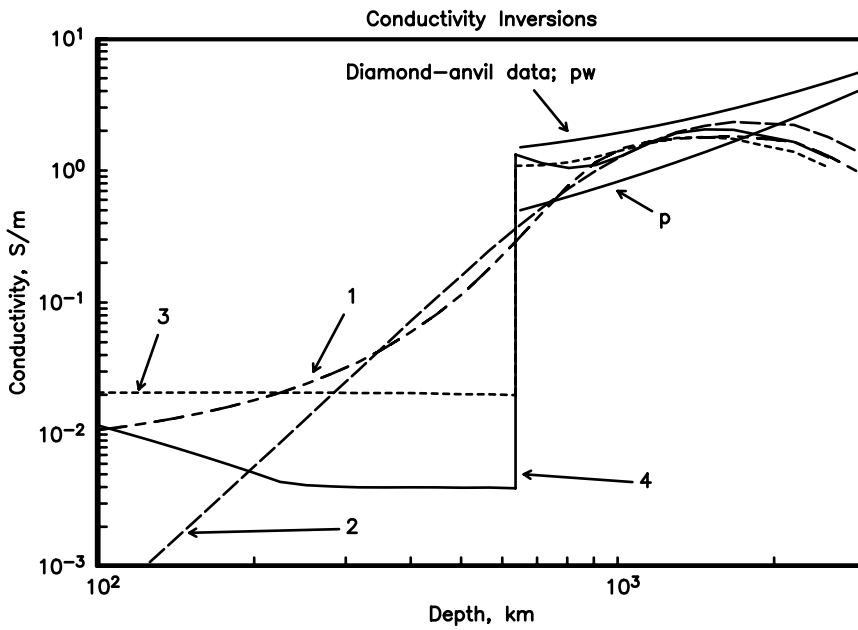


Figure 6.3.2

