6. Gibbs' Phenomenon

We keep meeting this issue and it is unsatisfactory for you not to have a treatment, particularly since Riley et al. duck the problem too. So I will give you a brief tour. The question revolves around any Fourier series expansion of a function with a (finite number of) simple jump discontinuities. We saw them in Figure 2, page 6 because the function $f(x) = x$ is not continuous on a circle, and so there is an effective jump between $x = -\pi$ and $x = \pi$. Here I examine the simplest case. We work on the interval $(0, +1)$ and use the complex basis $e_n(x) = \mathrm{e}^{2\pi i n x}$ with $n = 0, \pm 1, \pm 2, \cdots$ which is complete, and we have $\| e_n \| = 1$.

First we write out the expansion of a function f, taking the terms out to $|n| = N$:

$$S_N(f) = \sum_{n=-N}^{N} c_n e_n(x) = \sum_{n=-N}^{N} (f, e_n)\, e_n(x) \tag{6.1}$$

$$= \sum_{n=-N}^{N} \left(\int_0^1 f(y)\, \mathrm{e}^{-2\pi i n y}\, dy \right) \mathrm{e}^{2\pi i n x} \tag{6.2}$$

$$= \int_0^1 dy \left(f(y) \sum_{n=-N}^{N} \mathrm{e}^{2\pi i n (y-x)} \right) = \int_0^1 dy\, f(y)\, D_N(y-x)\,. \tag{6.3}$$

The integral is an example of a **convolution,** an operation we will see more of with Fourier transforms. We can evaluate the function D_N:

$$D_N(u) = \sum_{n=-N}^{N} \mathrm{e}^{2\pi i n u} = \mathrm{e}^{-2\pi i \mathrm{N}} \sum_{n=0}^{2N} \mathrm{e}^{2\pi i n u}\,. \tag{6.4}$$

Suppose we set $w = \mathrm{e}^{2\pi i u}$. Then (6.4) becomes very simple, a geometric series, $1 + w + w^2 + \cdots w^{2N}$, which we can sum exactly:

$$D_N(u) = w^{-N}(1 + w + w^2 + \cdots + w^{2N}) = w^{-N} \left(\frac{w^{2N+1} - 1}{w - 1} \right) \tag{6.5}$$

$$= \frac{w^{N+½} - w^{-N-½}}{w^{½} - w^{-½}} \tag{6.6}$$

$$= \frac{\sin \pi(2N+1)u}{\sin \pi u}\,. \tag{6.7}$$

In the present context this function is called the **Dirichlet kernel.** We are interested in $D_N(x)$ for large values of N, when we have taken a lot of terms in (6.1). A graph of the Dirichlet kernel is shown in Figure 5, with $N = 25$.

Returning to (6.3), let us now choose a particular function for f. We will treat the unit step, or **Heavisde function**, in this case at $x = ½$: let

$$f(x) = H(x - ½) = \begin{cases} 0, & x < ½ \\ 1, & x \geq ½ . \end{cases} \tag{6.8}$$

Let us call the sum for this function $Q_N(x)$; then

$$Q_N(x) = \int_0^1 H(y - ½)\, D_N(y)\, dy = \int_{½}^{1} D_N(y - x)\, dy = \int_{-1+x}^{x-½} D_N(t)\, dt \tag{6.9}$$

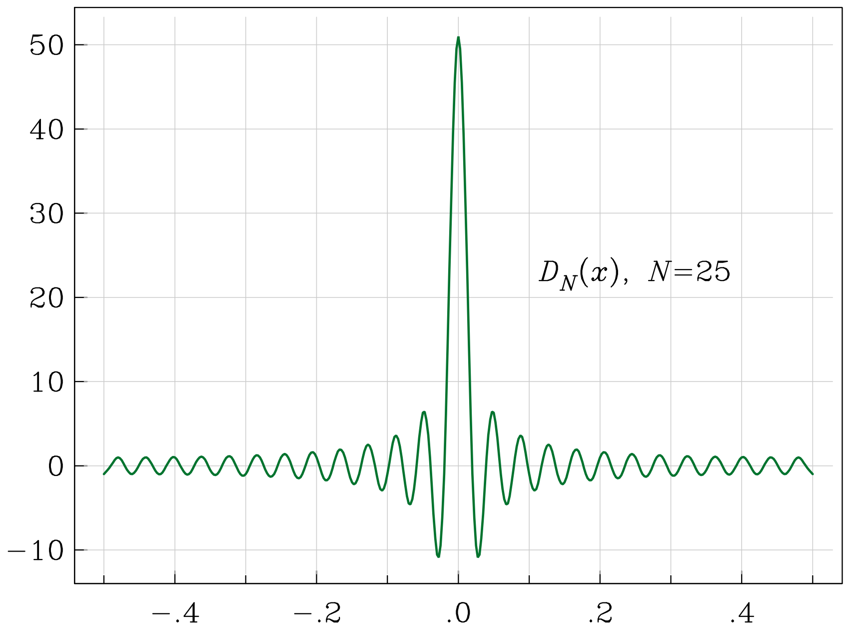
since $D_N(t) = D_N(-t)$. We are interested in the behavior near the jump at $x = ½$, so we write $x = \xi + ½$ and

$$Q_N(\xi + ½) = \int_{-½+\xi}^{\xi} D_N(t)\, dt = \int_{-½+\xi}^{\xi} \frac{\sin(2N+1)\pi t}{\sin \pi t}\, dt . \tag{6.10}$$

We see from the figure that all the interesting behavior occurs from small ξ; it is time to make a few approximations: near $\xi = 0$ we can write $\sin \pi t = \pi t$ and we suppose N is large that we can replace $2N + 1$ with $2N$.

$$Q_N(\xi + ½) \sim \frac{1}{\pi} \int_{-\pi N}^{2N\pi\xi} \frac{\sin s}{s}\, ds \sim \frac{1}{\pi} \int_{-\infty}^{2N\pi\xi} \frac{\sin s}{s}\, ds \tag{6.13}$$

Figure 5: The Dirichlet kernel



$$= = ½ + \frac{1}{\pi}\int_0^{2N\pi\xi} \frac{\sin s}{s}\, ds \tag{6.14}$$

I have also used the fact that the integral on $(0,\infty)$ is $½\pi$ something we will show later on. The integral of $\sin t/t$ is called $si(t)$ and is a commonly used function – in MATLAB it is called sinint. I plot (6.14) in Figure 6 below. Notice how the result depends on the number of terms N only on the way the x axis is scaled. As N becomes large the picture is compressed in x, but its fundamental shape never changes: there is always an overshoot and an undershoot of the same magnitude, which turns out to be about 8.94 percent. This means that the maximum difference between the original step function and the sum of the series does not tend to zero with increasing N; in other words the Fourier series $S(f)$ is not uniformly convergent for this function.

The same analysis applies at any jump discontinuity, not just the unit step, and not just at $x = ½$.

Figure 6: Gibbs' overshoot function.