The Addition Theorem and Coordinate Rotation

Here are some notes to clarify what I said about the Addition Theorem being a special case of the a more general coordinate rotation. Suppose we refer all the $Y_l^m(\theta, \phi)$ to a particular fixed coordinate system with north pole along $\hat{\mathbf{z}}$. Let the unit vector $\hat{\mathbf{u}}$ be fixed too, but allow $\hat{\mathbf{s}}$ to move around. Then we can view the Spherical Harmonic Addition Theorem

$$P_l(\hat{\mathbf{s}} \cdot \hat{\mathbf{u}}) = \sum_{m=-l}^{l} \frac{4\pi}{2l+1} Y_l^m(\hat{\mathbf{s}}) Y_l^m(\hat{\mathbf{u}})^* \tag{1}$$

as a statement that the function of $\hat{\mathbf{s}}$ on the left has a spherical harmonic expansion of the form

$$f(\hat{\mathbf{s}}) = P_l(\hat{\mathbf{s}} \cdot \hat{\mathbf{u}}) = \sum_{m=-l}^{l} c_m Y_l^m(\hat{\mathbf{s}}) \tag{2}$$

where the coefficients are simply

$$c_m = \frac{4\pi}{2l+1} Y_l^m(\hat{\mathbf{u}})^* . \tag{3}$$

But now, suppose that we view the vector $\hat{\mathbf{u}} = \hat{\mathbf{z}}'$ as the z axis of another system of coordinates. We know that in that system, the function $f(\hat{\mathbf{s}})$ in (2) in in fact a spherical harmonic of degree l and order zero. In that coordinate system we have that

$$f(\hat{\mathbf{s}}) = \left(\frac{2l+1}{4\pi}\right)^{1/2} Y_l^0(\hat{\mathbf{s}})' \tag{4}$$

where the prime $()'$ means the coordinates θ and ϕ are measured in the primed coordinate system. Of course the transformation from z to z' is a rotation. So (2) can be written

$$Y_l^0(\hat{\mathbf{s}})' = \sum_{m=-l}^{l} \left[c_m \left(\frac{2l+1}{4\pi}\right)^{1/2} \right] Y_l^m(\hat{\mathbf{s}}) = \sum_{m=-l}^{l} C_m Y_l^m(\hat{\mathbf{s}}) \tag{5}$$

which says how the order-zero spherical harmonic function referred to the z' system looks when it is referred to the z coordinate system.