9. Spherical Harmonics

Now we come to some of the most ubiquitous functions in geophysics, used in gravity, geomagnetism and seismology. Spherical harmonics are the Fourier series for the sphere. These functions can are used to build solutions to Laplace's equation and other differential equations in a spherical setting.

We shall treat spherical harmonics as eigensolutions of the surface Laplacian. This would be like developing Fourier series as eigensolutions of the operator $(d/dx)^2$ on a finite line, but with boundary conditions that y and dy/dx match at the two ends. We sometimes get some mileage from representing a thing in two ways, one within a fixed coordinate system, the other in coordinate-free form. First we need a spherical polar coordinate system: see the figure. The origin O is always fixed to be the center of the unit sphere, and all coordinates are referred to that origin. Let us define a **surface gradient** for the sphere in two ways:

$$\nabla_1 = \hat{\theta}\,\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{\sin\theta}\,\frac{\partial}{\partial\phi} \tag{1}$$

$$= r\nabla - \mathbf{r}\,\frac{\partial}{\partial r}\ . \tag{2}$$

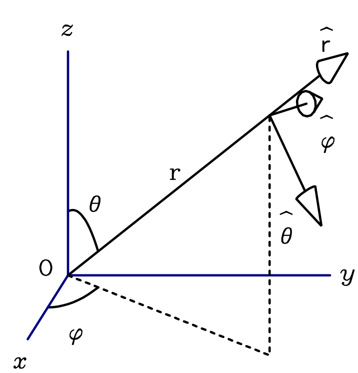
The subscript one is to remind us the operator acts over the unit sphere, $S(1)$. The first definition shows how to compute the surface gradient in a spherical polar coordinate system; the second assumes the function is defined in all of space and just subtracts out the radial part. The second definition shows the operator is independent of coordinate orientation and also that nothing funny happens at the poles. The ordinary Laplacian operator in $\mathbb{R}^3$ is

$$\nabla^2 = \frac{\partial}{\partial x_i}\,\frac{\partial}{\partial x_i} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{3}$$

$$= \frac{1}{r}\,\frac{\partial^2}{\partial r^2}\,r + \frac{1}{r^2}\,\nabla_1^2 \tag{4}$$

where ∇_1^2 is the **surface Laplacian**, sometimes also called the *Beltrami operator*;

Figure 9: The spherical polar coordinate system



relative to a coordinate system

$$\nabla_1^2 = \frac{\partial^2}{\partial\theta^2} + \cot\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \,. \tag{5}$$

We can think of ∇_1^2 as the ordinary Laplacian, with the radial part subtracted and scaled by r^2 to make it unitless: from (4)

$$\nabla_1^2 = r^2\nabla^2 - r\,\frac{\partial^2}{\partial r^2}\,r \,. \tag{6}$$

As with (2), this equation shows the operator is independent of any coordinate system and the singularity at the poles is not intrinsic to ∇_1^2, but is an artifact of the coordinates. We remark there is another surface operator $\nabla_s = r^{-1}\nabla_1$; this one has dimensions of $1/L$ like d/dx or the regular gradient operator ∇.

To develop spherical harmonics we ask for the eigenvalues and eigenfunctions of the surface Laplacian. We do this in part because, just as in $\mathbb{R}^3$ the eigenvectors of a symmetric matrix provide an orthogonal basis for the space, so here the self adjoint operator has a collection of orthogonal functions that span the function space. We regard complex-valued functions on the unit sphere as elements in the Hilbert space complex $L_2(S(1))$; here $S(1)$ is shorthand for the unit sphere, the set of points $|\mathbf{r}| = 1$. Then the norm is

$$\|f\| = \left(\int_{S(1)} d^2\hat{\mathbf{s}}\; |f(\hat{\mathbf{s}})|^2 \right)^{1/2} \tag{7}$$

and the notation $d^2\hat{\mathbf{s}}$ is a surface element on $S(1)$, which would be $\sin\theta\, d\theta\, d\phi$ in a particular polar coordinate system. We find it handy to work complex functions on $S(1)$. The space $L_2(S(1))$ comes with the inner product

$$(f,\, g) = \int_{S(1)} d^2\hat{\mathbf{s}}\; f(\hat{\mathbf{s}})\, g(\hat{\mathbf{s}})^* \tag{8}$$

It can be show that the service Laplacian is self adjoint, that is

$$(\nabla_1^2 f,\, g) = (f,\, \nabla_1^2 g)\,. \tag{10}$$

Furthermore, as we have already remarked, the eigenfunctions of self adjoint operators are orthogonal.

Suppose $u = u(\hat{\mathbf{r}}) = u(\theta, \phi)$ satisfies the eigenvalue equation

$$\nabla_1^2 u = \lambda\, u \tag{11}$$

and u is continuous everywhere on $S(1)$ but is not identically zero, then u is an eigenfunction for ∇_1^2. Not every value of λ can support such solutions. Only when λ is one of the integers

$$\lambda = 0, -2, -6, -12, -20, \cdots - l(l+1) \cdots \tag{12}$$

does (11) have nontrivial solutions. How does one prove this? The traditional way is by a technique called *separation of variables*, assuming that u can be written as a product of two single-argument functions: $u(\theta, \phi) = \Theta(\theta)\,\Phi(\phi)$, substituting in and getting two one-dimensional eigenvalue problems, one each for Θ and Φ. See Morse and Feshbach, *Methods of Mathematical Physics*, Vol II, 1953, for example. Chapter 3 of Backus, Parker and Constable (*Foundations of Geomagnetism*, 1996) does this entirely differently, by looking at homogeneous harmonic polynomials.

We call l the **degree** of the spherical harmonic. The eigenfunctions of ∇_1^2 associated with the eigenvalues are called **spherical harmonics**; we write them

$$u(\theta,\phi) = Y_l^m(\theta,\phi)\,. \tag{13}$$

We will give an explicit formula for these functions later; they are complex-valued on the sphere. The eigenvalues in (12) are not simple (except $\lambda=0$), but $2l+1$-fold **degenerate**. This means that associated with the eigenvalue $\lambda=-6=-2\times(2+1)$, say, there are $5=2\times2+1$ different (that is, linearly independent) eigenfunctions. This is where the index m comes in: for each degree l the **order** m can be any one of $-l, -l+1, \cdots 0, 1, \cdots l$, giving the required number of different functions for any l. In the traditional arrangement these are chosen to be mutually orthogonal. Thus the spherical harmonics form an orthogonal family:

$$(Y_l^m, Y_n^k) = \int_{S(1)} d^2\hat{\mathbf{s}}\; Y_l^m(\hat{\mathbf{s}})\, Y_n^k(\hat{\mathbf{s}})^* = 0, \quad m \neq k \text{ or } l \neq n\,. \tag{14}$$

We usually scale the spherical harmonics to be of unit norm:

$$\|\, Y_l^m \,\| = 1 \tag{15}$$

then the spherical harmonics are said to be **fully normalized**, although not everyone does this. With fully normalized harmonics (14) and (15) combine to give

$$(Y_l^m, Y_n^k) = \delta_{ln}\, \delta_{mk}\,. \tag{16}$$

Notice that any linear combination of eigenfunctions of degree l is also a valid eigenfunction with eigenvalue $-l(l+1)$.

It is time to write out an explicit form for Y_l^m. These solutions are the ones obtained by the separation of variables mentioned earlier – they are each a product of a function of θ (colatitude) and one of ϕ (longitude). Here we go:

$$Y_l^m(\theta,\phi) = N_{lm}\, \mathrm{e}^{\mathrm{i}m\phi}\; P_l^m(\cos\theta) \tag{17}$$

where N_{lm} is a normalization constant to adjust the size of the functions; as I mentioned earlier, I usually choose it to enforce (15); then

$$N_{lm} = (-1)^m \left(\frac{2l+1}{4\pi}\right)^{1/2} \left(\frac{(l-m)!}{(l+m)!}\right)^{1/2}. \tag{18}$$

There are however several different conventions regarding N_{lm}. For example, leaving off the alternating sign and removing the factor $((2l+1)/4\pi)^{1/2}$, results in functions that are *Schmidt normalized*. Our convention, (18), is most convenient for theoretical work, because of (15), which the others fail to comply with. The factor $\exp(im\phi)$ is just the complex Fourier basis for functions of longitude on complex $L_2(-\pi,\pi)$. Many old-fashioned authors (in geomagnetism especially) use sines and cosines with real coefficients here instead. The last factor is called an **Associated Legendre function** and is defined by

$$P_l^m(\mu) = \frac{1}{2^l l!} (1-\mu^2)^{m/2} \frac{\partial^{l+m}}{\partial \mu^{l+m}} (\mu^2 - 1)^l \ . \tag{19}$$

When the order $m=0$, the Associated Legendre function becomes a polynomial in μ and instead being written $P_l^0(\mu)$ it is designated $P_l(\mu)$, the **Legendre polynomial** which have seen several time already. We provide a very short table. Here $s = \sin\theta = (1-\mu^2)^{1/2}$.

$$P_0(\mu) = 1$$

$$P_1(\mu) = \mu \qquad P_1^1(\mu) = s$$

$$P_2(\mu) = (3\mu^2 - 1)/2 \qquad P_2^1(\mu) = 3\mu s \qquad P_2^2(\mu) = 3s^2$$

$$P_3(\mu) = \mu(5\mu^2 - 3)/2$$

$$P_3^1(\mu) = 3s(5\mu^2 - 1)/2$$

$$P_3^2(\mu) = 15s^2\mu \qquad P_3^3(\mu) = 15s^3$$

$$P_4(\mu) = (35\mu^4 - 30\mu^2 + 3)/8$$

$$P_4^1(\mu) = 5s\mu(7\mu^2 - 3)/2$$

$$P_4^2(\mu) = 15s^2(7\mu^2 - 1)/2$$

$$P_4^3(\mu) = 105s^3\mu \qquad P_4^4(\mu) = 105s^4$$

$$P_5(\mu) = \mu(63\mu^4 - 70\mu^2 + 15)/8$$

$$P_5^1(\mu) = 15s(21\mu^4 - 14\mu^2 + 1)/8$$

$$P_5^2(\mu) = 105s^2\mu(3\mu^2 - 1)/2$$

$$P_5^3(\mu) = 105s^3(9\mu^2 - 1)/2$$

$$P_5^4(\mu) = 945s^4\mu \qquad P_5^5(\mu) = 945s^5.$$

We have listed only positive m since there is a nice symmetry that allows us to do without:

$$Y_l^{-m} = (-1)^m (Y_l^m)^* \ . \tag{20}$$

We also note these special values – they are worth remembering:

$$P_l(1) = 1, \quad P_l^m(1) = 0,\ m \neq 0, \quad P_l^l(\cos\theta) = c_l \sin^l\theta \ . \tag{21}$$

Next we note one of the most important properties of the spherical harmonics: they are a **complete** set for expanding functions on $L_2(S(1))$. Thus in the usual way, when

$$f(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{lm} Y_l^m(\theta,\phi) \tag{22}$$

we get the coefficients from

$$c_{lm} = (f, Y_l^m) = \int_{S(1)} d^2\hat{\mathbf{s}}\; f(\hat{\mathbf{s}})\, Y_l^m(\hat{\mathbf{s}})^* \; . \tag{23}$$

It is this property that makes spherical harmonics so useful. Orthogonality is a property that follows from the self-adjointness of ∇_1^2. Completeness follows from a more subtle property, that the inverse operator of ∇_1^2 is compact, a property that would take us too far afield to explore.

Unless you have seen them before, you probably have no idea what the spherical harmonics look like at this point. One further property of the Associated Legendre functions helps: for $-1 < \mu < 1$ the function $P_l^m(\mu)$ crosses zero exactly $l - m$ times. With this information we can picture the Y_l^m on a sphere, by graphing the places where Re Y_l^m is zero; Im Y_l^m has the same pattern, but rotated about the $\hat{\mathbf{z}}$ axis, as can be seen at once from (17). So here is the recipe: for Y_l^m there are two sets of lines on the sphere where Re Y_l^m vanishes: (a) a set of $2m$ equally-spaced halves of great circles through the poles (meridians) coming from the exponential; (b) a set of $l - m$ small circles with planes normal to the z coordinate axis. See the pictures in Section 7. For example, when $m = 0$ we always have circular symmetry about the z axis.

After we have drawn a few we soon get the idea that the higher degree harmonics are shorter wavelength than the lower ones. In fact one can consider the surface harmonics to be standing waves in the surface of the sphere. **Jeans' formula** says on a unit sphere

$$\lambda(l) = \frac{2\pi}{l + ½} \tag{24}$$

where $\lambda(l)$ is the wavelength of the waves of degree l. Notice this is *independent of* m and the position on the sphere. In analogy with Fourier filtering, people often make a spherical harmonic expansion, then remove the high-degree terms in order to emphasize the long-wavelength features; or conversely, the low degree-terms are removed to show the high-wavenumber behavior, of the geoid, for example. Parseval's Theorem for a spherical harmonic expansion (22) gives

$$||f||^2 = \sum_{l=0}^{\infty} \left[\sum_{m=-l}^{l} |c_{lm}|^2 \right] \tag{25}$$

The terms in square brackets gives a **power spectrum** of f as a function of reciprocal wavelength or wavenumber (l is sometimes call the spherical wavenumber). This is a powerful way of showing how a geophysical field over a sphere is distributed according to its length scales.

Two powerful properties of the spherical harmonics that do not follow immediately from their connection with ∇_1^2 are the Addition Theorem and the generating function for Legendre polynomials. We state the **generating function** first:

$$\frac{1}{(1 - 2\mu x + x^2)^{½}} = \sum_{l=0}^{\infty} x^l \; P_l(\mu) \; . \tag{26}$$

We can prove this later from the development of the potential of a point mass.

The **Spherical Harmonic Addition Theorem** says

$$P_l(\hat{\mathbf{u}} \cdot \hat{\mathbf{v}}) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_l^m(\hat{\mathbf{u}})\, Y_l^m(\hat{\mathbf{v}})^* \,. \tag{27}$$

The following proof won't be found in any of the books; for another see *Foundations of Geomagnetism*, Chapter 3. Consider a delta function on $S(1)$ which peaks at the point $\hat{\mathbf{u}}$ which we will write as $\delta(\hat{\mathbf{r}} - \hat{\mathbf{u}})$. From our discussion of the Reproducing Kernel (Section 8) when $N \to \infty$ the function K_N becomes a delta function. Taking the complex nature of Y_l^m into account, we find that (8.1) gives us the SH expansion of the delta function at $\hat{\mathbf{u}}$:

$$\delta(\hat{\mathbf{s}} - \hat{\mathbf{u}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_l^m(\hat{\mathbf{s}})\, Y_l^m(\hat{\mathbf{u}})^* \,. \tag{28}$$

Next consider this expansion for $\hat{\mathbf{u}} = \hat{\mathbf{z}}$, the north pole. The function is symmetric about the z axis, so all the coefficients in (28), vanish except those with $m = 0$. The same result follows from the second member of (21) which implies that $Y_l^m(\hat{\mathbf{z}}) = Y_l^m(½\pi, \phi) \ = N_{lm} \mathrm{e}^{im\phi} P_l^m(1) = 0$ when $m \neq 0$. Thus

$$\delta(\hat{\mathbf{s}} - \hat{\mathbf{z}}) \ = \sum_{l=0}^{\infty} Y_l^0(\hat{\mathbf{s}}) Y_l^0(\hat{\mathbf{z}})^* = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi}\, P_l(\cos\theta) \tag{30}$$

$$= \sum_{l=0}^{\infty} \frac{2l+1}{4\pi}\, P_l(\hat{\mathbf{z}} \cdot \hat{\mathbf{s}}) \,. \tag{31}$$

Next consider the delta function peak to be moved to some other point on the unit sphere, say $\hat{\mathbf{u}}$; then because (31) does not depend on the fact that $\hat{\mathbf{z}}$ is a coordinate axis, we can replace $\hat{\mathbf{z}}$ by $\hat{\mathbf{u}}$ and the equation is still true:

$$\delta(\hat{\mathbf{s}} - \hat{\mathbf{u}}) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi}\, P_l(\hat{\mathbf{u}} \cdot \hat{\mathbf{s}}) \,. \tag{32}$$

Equations (28) and (31) are expansions of the same function. We can compare these two expressions degree by degree. Since these are both expansions in the eigenfunctions of ∇_1^2 and the eigenfunctions of different degrees are orthogonal to each other, the degree-l terms in each sum must be equal to each other: hence (27).

Here are a couple of useful spherical harmonic expansions. We combine the generating function and the Addition Theorem together, to find an expansion for the potential of point mass, not at the origin. With $s < r$ we write

$$\frac{1}{R} = \frac{1}{|\,\mathbf{r} - \mathbf{s}\,|} = \frac{1}{(r^2 + s^2 - 2rs\, \hat{\mathbf{r}} \cdot \hat{\mathbf{s}})^{½}} \tag{33}$$

$$= \frac{1}{r} \frac{1}{[1 + (s/r)^2 - 2(s/r)\, \hat{\mathbf{r}} \cdot \hat{\mathbf{s}}]^{½}} \,. \tag{34}$$

Now we recognize the generating function (26):

$$\frac{1}{R} = \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{s}{r}\right)^l P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{s}}) \tag{35}$$

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{s^l}{r^{l+1}} Y_l^m(\hat{\mathbf{r}}) Y_l^m(\hat{\mathbf{s}})^* \, . \tag{36}$$

The last line follows from the Addition Theorem. So if we fix $r = |\mathbf{r}|$ and $\mathbf{s}$ and imagine moving around on the sphere of radius r, (36) is the spherical harmonic expansion of the potential $1/R$ as seen on that sphere.

A similar expansion (which we won't derive) used in scattering theory is the following. Suppose a plane wave comes along and hits a sphere. The wave field over the sphere can be decomposed in a spherical harmonic expansion:

$$e^{2\pi i\mathbf{k}\cdot\mathbf{r}} = (2\pi)^{3/2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} e^{i\pi l/2} \frac{J_{l+\frac{1}{2}}(2\pi kr)}{(2\pi kr)^{\frac{1}{2}}} Y_l^m(\hat{\mathbf{r}}) Y_l^m(\hat{\mathbf{k}})^* \tag{37}$$

where $J_{l+\frac{1}{2}}$ is a **Bessel function**, an important kind of function we will discuss later on in Fourier transforms.

We mention the fact that the surface gradients of spherical harmonics are also orthogonal and complete in a certain sense. They can be used to expand surface vector fields, that is vectors defined in $S(1)$ that are always tangent to the sphere. The relevant relation is

$$\int_{S(1)} d^2\hat{\mathbf{s}} \, \nabla_1 Y_l^m(\hat{\mathbf{s}}) \cdot \nabla_1 Y_n^k(\hat{\mathbf{s}})^* = l(l+1)\, \delta_{ln} \delta_{mk} \tag{38}$$

which we can interpret an an inner product on a different kind of Hilbert space on $S(1)$.

We mention briefly two more advanced topics. First, the 3-j symbols. Occasionally one runs into the need to perform integrals over the sphere of products of three spherical harmonics. These can always be done simply, and there are symmetry relationships that make many such products vanish. The problem was first studied systematically in quantum mechanics of angular momentum, and so the reference everyone uses is: Edmonds, *Angular Momentum in Quantum Mechanics*. There is a table made worthless because if its lack of supporting explanation in Chap 27 Abramowitz and Stegun, *Handbook of Mathematical Functions*, Dover, 1970. This book is worth knowing about in general, and has many results on Associated Legendre functions in Chaps 8 and 22.

And second, one may require a given spherical harmonic expansion in a rotated coordinate system; how does the spherical harmonic $Y_l^m(\mathbf{r})$ transform to a different set of coordinate axis? Obviously (perhaps) the new Y_l^m will be a linear combination of harmonics of the same degree, a sum over m only. But what are the coefficients? The Addition Theorem can be viewed as a special case. The general answer is complicated. See Winch's article in in the Encyclopedia of Geophysics for this and much more.

A Table of Spherical Harmonic Lore

	Property	Formula	Comments
1.	Laplacian in polar coordinates	$\nabla^2 = \frac{1}{r^2}\nabla_1^2 + \frac{1}{r}\frac{\partial^2 r}{\partial r^2}$	∇_1^2 is angular part of familiar Laplacian
2.	Eigenvalue	$\nabla_1^2 Y_l^m = -l(l+1)Y_l^m$, $l = 0, 1, 2, \cdots$	There are $2l+1$ linearly independent eigenfunctions per l
3.	Orthogonality	$\int d^2\hat{\mathbf{s}}\; Y_l^m(\hat{\mathbf{s}})\, Y_n^k(\hat{\mathbf{s}})^* = 0$, unless $l = n$ and $m = k$	True for every normalization
4.	Theoretician's normalization	$\int d^2\hat{\mathbf{s}}\; \lvert Y_l^m(\hat{\mathbf{s}})\rvert^2 = 1$	Other choices: 4π or $4\pi/(2l+1)$
5.	Completeness	$f(\hat{\mathbf{s}}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_{lm} Y_l^m(\hat{\mathbf{s}})$	Works for any reasonable function f on $S(1)$
6.	Expansion coefficients	$c_{lm} = \int d^2\hat{\mathbf{s}}\; f(\hat{\mathbf{s}})\, Y_l^m(\hat{\mathbf{s}})^*$	Requires property 4
7.	Addition Theorem	$\frac{2l+1}{4\pi} P_l(\hat{\mathbf{s}}\cdot\hat{\mathbf{r}}) = \sum_{m=-l}^{l} Y_l^m(\hat{\mathbf{s}})\, Y_l^m(\hat{\mathbf{r}})^*$	Requires property 4
8.	Wavelength of Y_l^m	$\frac{2\pi}{l+½}$	Depends only on degree l, not on order m or $\hat{\mathbf{s}}$
9.	Appearance	Re Y_l^m vanishes on $2m$ meridians and $l-m$ parallels	Im Y_l^m the same but rotated about $\hat{\mathbf{z}}$
10.	Parseval's Theorem	$\int d^2\hat{\mathbf{s}}\; \lvert f(\hat{\mathbf{s}})\rvert^2 = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \lvert c_{lm}\rvert^2$	Requires property 4. Get RMS value of f by dividing by 4π and taking square root
11.	Generating function	$\frac{1}{(1-2\mu r + r^2)^{½}} = \sum_{l=0}^{\infty} r^l P_l(\mu)$	Often used in conjunction with property 7
12.	Another orthogonality	$\int d^2\hat{\mathbf{s}}\; \nabla_1 Y_l^m(\hat{\mathbf{s}})\cdot\nabla_1 Y_n^k(\hat{\mathbf{s}})^* = l(l+1)\,\delta_{ln}\,\delta_{mk}$	Requires property 4.

10. Application: the Solution of Laplace's Equation

This is the application of spherical harmonics that we need for potential theory. We consider the region inside a spherical shell $R_1 \le r \le R_2$ inside which Laplace's equation is obeyed by V, a potential:

$$\nabla^2 V = 0 \tag{1}$$

Sometimes we will let R_2 tend to infinity, but it is useful to keep it finite for now. Let us write V as a function of spherical polar coordinates $V(r, \theta, \phi)$ relative to O, the origin at the center of the concentric spheres. Provided V is reasonably well behaved we can write it thus:

$$V(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} V_{lm}(r)\, Y_l^m(\theta, \phi) \ . \tag{2}$$

This follows because in any spherical surface the function V can be expanded in surface harmonics, and (2) simply says there is a different set of expansion coefficient for each radius r. An expansion like (2) is perfectly general, and does not depend on V being harmonic. Let us now add the condition that V obeys Laplace's equation, which we write using (9.4):

$$\frac{1}{r} \frac{\partial^2 rV}{\partial r^2} + \frac{1}{r^2} \nabla_1^2 V = 0 \ . \tag{3}$$

Inserting (2) into (3) and recalling the eigenfunction property that

$$\nabla_1^2 Y_l^m = -\,l(l+1)\, Y_l^m \tag{4}$$

we obtain

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[\frac{1}{r} \frac{d^2 rV_{lm}}{dr^2} - \frac{l(l+1)}{r^2} V_{lm} \right] Y_l^m(\theta, \phi) = 0 \ . \tag{5}$$

Because of the orthogonality of the surface harmonics the only way that this sum can be zero is if the factor in square brackets vanishes identically for every r and each l and m separately. Thus we find the ordinary differential equations:

$$\frac{d^2 rV_{lm}}{dr^2} - \frac{l(l+1)}{r} V_{lm}(r) = 0 \ . \tag{6}$$

Notice the $2l+1$ different functions with the same l but differing m obey the same differential equation. The standard way of solving such equations is by substituting a power series. Omitting the details, we get

$$V_{lm}(r) = A\, r^l + \frac{B}{r^{l+1}} \tag{7}$$

as the most general solution. Clearly each degree and order in (5) will generally be associated with different constants A and B; thus the general solution to (3) is given by

$$V(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(A_{lm} r^l + \frac{B_{lm}}{r^{l+1}} \right) Y_l^m(\theta, \phi) \tag{8}$$

where the constants A_{lm} and B_{lm} are coefficients in the spherical harmonic expansion of the potential; they must be determined by experiment or analysis for each different potential V.

There is a physical interpretation of the two kinds of coefficients. We see that the contribution to V from the A_{lm} grows with increasing radius r. This implies that the sources for A_{lm}, that is the matter in the gravitational case, lie in the region **outside** R_2, since V increases with proximity to the sources. Similarly B_{lm} is associated with matter **inside** the inner sphere, radius R_1. The sum of the A_{lm} terms is called the **external part** of V and the B_{lm} sum the **internal part**. If the expansion (8) describes the gravitational field of the Earth, with O at its center, there can be no exterior part, as the is matter in question all lies within $S(R_1)$ Then it is conventional to rewrite (8) thus

$$V = -\frac{Gm_E}{r}\left[1 + \sum_{l=2}^{\infty}\sum_{m=-l}^{l} c_{lm}\left(\frac{a}{r}\right)^l Y_l^m(\theta,\phi)\right]. \tag{9}$$

The numbers c_{lm} are sometimes called **Stokes coefficients**. Notice the absence of $l=1$ terms because O is at the center of mass so there is no dipole term. Also note how the c_{lm} are dimensionless, as the radial terms are scaled by the so-called reference radius a. As we mentioned earlier, a is often the equatorial radius, because then the inner sphere encloses all the matter; sometimes the Earth's mean radius is used instead (often people forget to say!). Of course, practical models of the Earth's potential do not sum to infinity, but very large degree model have been found with $l_{\max} = 360$; how many coefficients is that?

Books

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Morse, P. M. and H. Feshbach, *Methods of Theoretical Physics,* McGraw-Hill, New York, 1953.
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