FOURIER TRANSFORMS

Class Notes by Bob Parker

1. What is the Fourier Transform?

For a suitable function f of a single real variable the definition of the **Fourier transform** is:

$$\hat{f}(\nu) = \mathcal{F}[f] = \int_{-\infty}^{\infty} dt\ \mathrm{e}^{-2\pi i\nu t} f(t)\,. \tag{1.1}$$

This operation takes a real function and generates from it another complex-valued function, that is, a function of a single variable (here ν) that has a real part and imaginary part. We can think of this as a kind of linear mapping: a function goes in, and another (complex) function comes out.

It is easier to start with the idea of synthesizing a complex function f rather than a real one. What does the transform mean? In loose physical terms the Fourier transform (we'll write FT, from now on) produces a complex amplitude spectrum that shows how large the amplitude of the cosine and sine functions must be at each frequency ν, in a decomposition of the function into periodic parts, parts that each have a definite frequency. When the independent variable is a coordinate is space, **wavenumber**, which we will write as k, is more appropriate. Imagine building up the function f from (an infinite) sum of sines and cosines; this may sound implausible for an infinite interval, and indeed, not every function can be built like this, but many can. Because we must allow every possible frequency of sine and cosine, the sum must be an integral, which we would write:

$$f(t) = \int_{-\infty}^{\infty} d\nu\ \mathrm{e}^{+2\pi i\nu t} \hat{f}(\nu)\,. \tag{1.2}$$

For a particular frequency ν, the contribution to the sum is just

$$\mathrm{e}^{2\pi i\nu t} \hat{f}(\nu) = (\cos 2\pi\nu t + i \sin 2\pi\nu t)\hat{f}(\nu)\,. \tag{1.3}$$

You can see this function is a sine wave, but it has real and imaginary parts. As a function of t, the function in (1.3) repeats itself exactly, with **period** $1/\nu$; for functions in space the corresponding quantity is of course the **wavelength**, $\lambda = 1/k$. If the magnitude of the amplitude function $|\hat{f}(\nu)|$ is large at some frequency ν_0 compared to that at any frequency, we would expect $f(t)$ built by (1.2) to approximate a complex cosine wave with frequency ν_0. As an illustration, consider the function $\hat{g}$ which is a Gaussian hump with its peak at ν_0 shown in Figure 1a:

$$\hat{g}(\nu) = \mathrm{e}^{-(\nu-\nu_0)^2/2\sigma^2} \tag{1.4}$$

where σ is a standard measure of the width. If we plug this into (1.2) we get (by methods we shall describe later on)

$$g(t) = \sqrt{2\pi}\sigma \mathrm{e}^{-2\pi^2\sigma^2 t^2}\ \mathrm{e}^{2\pi i\nu_0 t}\,. \tag{1.5}$$

You see g is a product of two exponential factors, a Gaussian hump of width $1/2\pi\sigma$, and a complex cosine term with frequency ν_0. As σ becomes small, and the amplitude of $\hat{g}$ is more and more concentrated at ν_0, g looks more and more like a pure complex cosine, since the width of the Gaussian factor grows larger.

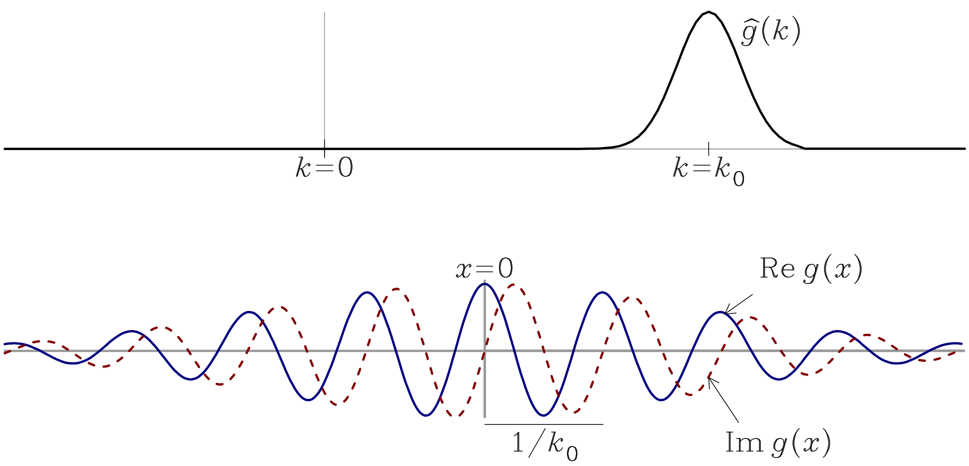
If you accept the plausibility of the idea that you can build a non-periodic function from a sum of periodic parts, the next question is, Given a function f, how do we find the corresponding amplitude function $\hat{f}$? Of course, the answer has already been provided by (1.1). Remarkably, the amplitude calculating formula, (1.1) is almost the same as the synthesizing expression, (1.2). The only difference is the sign change in the exponential. Equation (1.2) is called the **inverse Fourier transform**, and can be written

$$f(t) = \mathcal{F}^{-1}[\hat{f}]. \tag{1.6}$$

Obviously, if you take the FT of a function and then take the inverse FT of the result, you should get back the function you started with. This is exactly true only for moderately well behaved functions, as we will discuss shortly.

A brief discussion of notation is in order. There are several slightly different conventions for the FT. I use the one with the factor 2π in the exponent, which has always been the engineering practice and has become the standard in the applied math literature. Riley et al. use a physicists' form with a $(2\pi)^{-1/2}$ outside. There are good reasons for preferring my notation, and I *strongly* recommend using it. First, there no other factors of 2π to remember in any of the other related results, the Convolution Theorem, Parseval's Theorem, etc., while all other notations have random factors; second, the parameter ν is a frequency, not an angular frequency in radians per second, and $1/\nu$ is a period or a wavelength, and so much easier to identify with a physical scale, not six times the scale. I use the notation $\hat{f}$ for the FT, which is common in the mathematical work, while Riley and company have $\tilde{f}$, which is

Figure 1a: The FT of a shifted Gaussian.



completely unheard of! Some authors will use F for the FT of f.

Two physical realizations of the FT that come to mind. The acoustical idea of decomposing a sound into its component frequencies, something suggested by musical tradition for sounds that are nearly periodic. If the pressure time series is $p(t)$, and its FT is $\hat{p}(\nu)$, then the magnitude squared $|\hat{p}(\nu)|^2 d\nu$ is the acoustical power (or energy, these terms seem to be interchangeable in this context) in the frequency band ν to $\nu + d\nu$; conventionally, we might use f or ν for frequency, not ν; for us f is unsuitable because I like to reserve this letter for the name of a function. A second familiar illustration comes from optics: the spectrum of a light source. Here we usually think of the intensity of the light as a function of wavelength, but a light signal in space $f(x)$ can be Fourier transformed and the magnitude squared $|\hat{f}(k)|^2$ is the spectrum as a function of wavenumber $k = 1/\lambda$. However, a proper model of light requires the concept of the FT of a random process, called the **power spectrum,** something we will be spending a lot of time on later in the course.

You will probably be troubled by the fact that, even when we start with a real function in (1.1), we get a complex one out (though not always, as we shall see). What does this mean? Looking again at the process of building up a function from its Fourier spectrum, equation (1.2), and thinking about all the functions that can be built by summing the real part, you will see that cosine functions of the form $\cos 2\pi\nu t$ with different real amplitudes can make only *even* functions of t. An even function must satisfy $F(t) = F(-t)$, but obviously not all functions are even. Similarly, sums of sines are always *odd* functions, with $F(t) = -F(-t)$. While in general a function is neither even nor odd, every real function can be written uniquely as a sum of an odd and an even part. So it turns out that the real part of the FT copes with the even part of the function and the imaginary part with the odd part; in general both parts are necessary. While on this topic, we note the following easily demonstrated symmetry: when $f(t)$ is real, then the real part of $\hat{f}(\nu)$ is always an even function, and the imaginary part is always odd.

Before going into further details we must ask, Under what conditions is (1.2) valid? It turns out things can go wrong in several ways: for example, the integral might not exist, or it might exist, but give the wrong answer. It is impractical to formulate a set of sufficient and necessary conditions on f guaranteeing the validity of the inverse formula. We can however come up with a lot of useful restrictions — sufficient conditions. For example, if f is smooth (having derivatives of arbitrary order) at every point, and it decays away faster than any negative power of t, then $\hat{f}$ has the same properties and (1.2) holds; the set of functions with these properties is called $\mathcal{S}$. The Gaussian function in (1.5) is a member of $\mathcal{S}$, for example. This is a very bland set of functions, however, and we often need to go outside it.

Another class, commonly used, which contains more lively functions is complex $L_2(-\infty, \infty)$, which we have met on a finite interval; now we need the set of functions for which

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 \, dt < \infty . \tag{1.7}$$

Notice the magnitude $|f|$ in the integrand, because f may be complex valued. The result here is that every f in L_2, that is obeying (1.7), has an FT $\hat{f}$ also in L_2. But now (1.2) is not precisely true, unless we agree to modify what is meant by equality of two functions! If we agree that two functions, f and g, are essentially the same if

$$||f - g|| = 0 \tag{1.8}$$

then f is essentially equal to $\mathcal{F}^{-1}[\hat{f}]$ for f in L_2. How could f and g differ in (1.8)? They can differ at points of discontinuity: for example, it could be that f is zero for all $t \le 0$, and is equal to $\cos t$ otherwise — a discontinuous function. If g vanishes for all $t < 0$ and is $\cos t$ when $t \ge 0$, then f and g differ only at one point, $t = 0$, where $f(0) = 0$ and $g(0) = 1$. But the integral in (1.8) cannot "see" the difference between such functions even though they are different. The set of all functions g that differ from a function f while (1.8) is true is called an **equivalence class.** Members of an equivalence class can be treated as single objects; here they are elements in the complete normed space L_2.

We can restore exact equality if we add further constraints, that f, in addition to (1.7), must be piecewise continuous and differentiable, and at any points of discontinuity we require

$$f(t) = \lim_{h=0} ½[f(t+h) + f(t-h)] . \tag{1.9}$$

In other words, at jumps, the function takes the mean value of its limiting values on either side. Members of $L_2(-\infty, \infty)$ with these additional properties have FTs in L_2 with the same properties.

But even this is not a big enough class, because sometimes we want to study functions that die away at infinity too slowly to fit in L_2, functions like $\sin t/|t|^{½}$. And sometimes we would like to take the FT of $\sin t$ itself, which leads to an asymmetric arrangement where the functions f belong to one class, and their transforms belong to another set of objects that are not functions at all! They are **distributions** or **generalized functions**, and are treated rigorously by using the idea of an equivalence class again, this time a class of sequences of functions. Riley et al. and Bracewell give a sloppy treatment of distributions (things like **delta functions**) which is perhaps useful as an introduction. I will not dwell on these issues, but refer you to the many books on this subject. The property that (1.2) is truly the inverse of (1.1) needs a proof — it is not obvious — but we will not provide one. And as we have hinted, the conditions under which (1.2) is valid, are the subject of a rich mathematical literature, which you should consult If you are interested in such things. For rigorous proofs of many things, see Dym, H. and H. P. McKean, *Fourier Series and Integrals,* 1972, or Körner, T. W., *Fourier Analysis,* 1988. I have mentioned these two books earlier in our discussion of the Fourier series. For a survey of the Fourier universe, but without proofs, see Champeney, D. C., *A Handbook of Fourier Theorems,* 1987. The all-time favorite for nonmathematicians is by Bracewell, R. N., *The Fourier Transform and its Applications,* 1986.

Exercises

1. Prove every real function of a real variable can be written uniquely as the sum of an even part and an odd part. Show the Fourier transform of a real, even

function is also real and even. What about the Fourier transform of a real, odd function?

2. Show that taking the Fourier transform of a function twice results in the original function, but with sign of the argument reversed:

$$\mathcal{F}[\mathcal{F}[f(t)]] = f(-t)\,.$$

What do you get if you Fourier transform a function four times? What do you deduce about the Fourier eigenvalue problem: $\mathcal{F}[\phi] = \lambda\phi$?

3. By taking the complex conjugate of (1.1) prove that for real function f

$$\mathcal{F}[f(-t)] = \hat{f}(\nu)^*$$

where $*$ denotes complex conjugation. Also show that if g is an even function

$$\mathcal{F}[g] = \mathcal{F}^{-1}[g]\,.$$

Does this result need g to be real?

Books

Most books covering Fourier transforms give a very watered down version of distribution theory, including Riley et al., and Bracewell. Keener's book gives a much better treatment. But the standard reference in applied mathematics remains

Bracewell, R. N., *The Fourier Transform and its Applications*, McGraw-Hill Book Co., 1986.
The favorite for engineers and nonmathematicians. Lots of pictures and helpful examples. In later editions Bracewell becomes obsessed with something called the Hartley transform!

Champeney, D. C., *A Handbook of Fourier Theorems,* Cambridge Univ. Press, 1987.
A thin book explaining the major theorems about Fourier transforms without going into their proofs. One of the few treatments in one book that deals with the analytical function theory and distributions in a truly rigorous way.

Dym, H. and H. P. McKean, *Fourier Series and Integrals,* Academic Press, New York, 1972.
Modern mathematical treatment, often terse, but in concise notation.

Körner, T. W., *Fourier Analysis*, Cambridge Univ, Press, 1988.
Rigorous yet approachable because of a clear intent in explain, but with a mathematical bias. Not very systematic in organization, more or less random, in fact.

Lighthill, M. J., *Fourier Analysis and Generalized Functions*, Cambridge Univ. Press, 1964.
A thin but dense treatment of Fourier transforms from the perspective of distribution theory (generalized functions as Lighthill calls them).

Keener, J. P., *Principles of Applied Mathematics*, Perseus Books, 1988.
Covers a lot of ground and shows the use of distributions in other contexts besides the Fourier setting.

2. More Properties, a Few Derived

The first property of the FT concerns what happens if we move the origin or scale the independent variable. Bracewell calls these the shift and the similarity theorems. They are trivial to prove: just make a change of variables in the integral and we find that, with real a:

$$\mathcal{F}[f(a(t+t_0))] = \frac{1}{|a|}\,\hat{f}\left(\frac{\nu}{a}\right)e^{2\pi i\nu t_0}\,. \tag{2.1}$$

Thus, translating the origin just introduces a multiplying complex exponential factor, while compressing a signal (which would mean making a large), stretches out the FT, and conversely. Another easily proved result that is very useful is a kind of converse of (2.1):

$$\mathcal{F}[f(t)\,e^{2\pi i\nu_0 t}] = \hat{f}(\nu-\nu_0)\,. \tag{2.2}$$

Recall from the exercise in the last section that *for real functions f*

$$\mathcal{F}[f(-t)] = \hat{f}(\nu)^* \tag{2.3}$$

and so combining this result with (2.1) gives

$$\hat{f}(-\nu) = \hat{f}(\nu)^* \tag{2.4}$$

but *only for real* $f(t)$. An important result for mathematicians is that the FT of a function in L_2 is also in L_2 and therefore the FT maps the space L_2 onto itself. In fact the result is even stronger: **the Fourier transform preserves the 2-norm**:

$$\int_{-\infty}^{\infty} dt\;|f(t)|^2 = \int_{-\infty}^{\infty} d\nu\;|\hat{f}(\nu)|^2 \quad\text{or}\quad \|f\| = \|\hat{f}\|\,. \tag{2.5}$$

This very important result appears under a variety of names. It is sometimes called the *Plancherel Identity* for Fourier transforms; Bracewell calls it *Rayleigh's Theorem*; to Riley et al. it is *Parseval's Theorem*. The result (2.5) is far from obvious, and we leave the proof to the experts. Recall the inner product for complex L_2:

$$(f,g) = \int_{-\infty}^{\infty} dt\; f(t)\; g(t)^* \tag{2.6}$$

Then it is also the case that the inner product is preserved under the FT:

$$(f,g) = (\hat{f},\hat{g})\,. \tag{2.7}$$

This can be proved from (2.5); see the Exercise at the end of the Section. Unfortunately, (2.7) is also called by some (eg, Champeney) **Parseval's Formula**, and to Bracewell it is the **Power Theorem**.

Suppose that we now take the derivative of f in (1.2) and assume that $\hat{f}$ is smooth and otherwise well behaved (say a member of $\mathcal{S}$), so that we can take the differential inside the integral; then

$$\frac{df}{dt} = \int_{-\infty}^{\infty} d\nu\;\frac{d}{dt}\,e^{+2\pi i\nu t}\hat{f}(\nu) \tag{2.8}$$

$$= \int_{-\infty}^{\infty} d\nu \; 2\pi i\nu \, e^{+2\pi i\nu t} \hat{f}(\nu) = \mathcal{F}^{-1}[2\pi i\nu \, \hat{f}] \, . \tag{2.9}$$

When we take the FT of this equation we see

$$\mathcal{F}\left[\frac{df}{dt}\right] = \mathcal{F}[\mathcal{F}^{-1}[2\pi i\nu \, \hat{f}]] = 2\pi i\nu \hat{f}(\nu) \, . \tag{2.10}$$

Thus the FT of the derivative is just a multiplication by $2\pi i\nu$ times the FT of the original function. This result is very useful for solving certain kinds of differential equations, because the differential equation becomes a simple algebraic equation. You will be able to convince yourself that differentiating again just multiplies by the factor $2\pi i\nu$ again. Equation (2.10) is in agreement with intuition about functions: when one differentiates a function, the resultant is rougher, more spiky, and more wiggly; but this is synonymous with the presence of more power at high frequencies. Our result says the same thing, but much more precisely: there is an amplification of the spectrum by the factor ν, which obviously grows linearly with increasing ν.

The same manipulations, but applied now to (1.1) give us the converse result:

$$\mathcal{F}[t \, f(t)] = -\frac{1}{2\pi i} \frac{d\hat{f}}{d\nu} \, . \tag{2.11}$$

So multiplying a function by t results in an FT that is, within a scalar factor, the derivative of the original FT.

Exercises

1. We need the complex inner product for FTs:

$$(f, g) = \int_{-\infty}^{\infty} f(t) g(t)^* \, dt \, .$$

Notice now that the inner product is not commutative: $(g, f) = (f, g)^*$; also note that while $(\alpha f, g) = \alpha(f, g)$, inner product $(f, \alpha g) = \alpha^*(f, g)$. From the norm invariance property (2.5) show that

$$(f, g) = (\hat{f}, \hat{g}) \, .$$

Hint: Consider $\| f + \lambda g \|^2$.

2. The classical cosine transform of the real function f is defined as

$$\mathcal{C}[f] = \int_{0}^{\infty} \cos 2\pi\nu t \; f(t) \, dt \, .$$

Show that this transform is its own inverse. What is the inverse of the sine transform, define in the analogous manner to $\mathcal{C}$?

A Short Table of Properties

	Name	Property	Comments
1	Scale and shift	$\mathcal{F}[f(a(t+t_0))] = \frac{1}{\lvert a \rvert}\hat{f}\left(\frac{\nu}{a}\right)e^{2\pi i\nu t_0}$	
2	Exponential factor	$\mathcal{F}[f(t)\,e^{2\pi i\nu_0 t}] = \hat{f}(\nu-\nu_0)$	
3	Double transform	$\mathcal{F}^2[f] = f(-t)$	
4	Time reversal	$\mathcal{F}[f(-t)] = \hat{f}(\nu)^*$	Real f
5	Freq reversal	$\hat{f}(-\nu) = \hat{f}(\nu)^*$	Real f
6	Norm preservation	$\lVert f \rVert = \lVert \hat{f} \rVert$	Known by many names
7	Inner prod preservation	$(f, g) = (\hat{f}, \hat{g})$	Complex inner prod
8	FT of derivative	$\mathcal{F}[df/dt] = 2\pi i\nu\hat{f}$	
9	Factor of t	$\mathcal{F}[t\,f(t)] = -\frac{1}{2\pi i}\frac{d\hat{f}}{d\nu}$	
10	Convolution	$\mathcal{F}[f*g] = \hat{f}\;\hat{g}$	See Section 4

3. Several Fourier Transforms

Before proceeding any further we should have before us a few simple FTs of actual functions. What appear at first sight to be the simplest examples, the FT of the sine or cosine, are in fact complicated, because the integrals in (1.1) diverge. Special arrangements need to be made to get an answer for such functions. For classical integration to converge, we need a function that dies away to zero for large t. Let us start with a function that decays exponentially:

$$f_1(t) = \begin{cases} e^{-at}, & t \ge 0 \\ 0, & t < 0\,. \end{cases} \tag{3.1}$$

In the following a, b, and c are always real positive numbers. Notice that this function is not smooth: it has a discontinuity at $t = 0$. We can perform the integral of (1.1) very easily:

$$\hat{f}_1(\nu) = \int_{-\infty}^{\infty} f_1(t)\, e^{-2\pi i\nu t}\, dt = \int_0^{\infty} e^{-(a+2\pi i\nu)t}\, dt\,. \tag{3.3}$$

Here the integrand is just $\exp(-\gamma t)$ for a constant complex γ with positive real part, and answer is quite elementary: $1/\gamma$. Hence

$$\hat{f}_1(\nu) = \frac{1}{a + 2\pi i\nu} \tag{3.4}$$

$$= \frac{a}{a^2 + 4\pi^2\nu^2} - i\,\frac{2\pi\nu}{a^2 + 4\pi^2\nu^2}\,. \tag{3.5}$$

Notice how the real part of $\hat{f}_1$ is even and the imaginary part is odd, as promised. Also notice how much more slowly $|\hat{f}|$ decays to zero than $|f|$. Why is this?

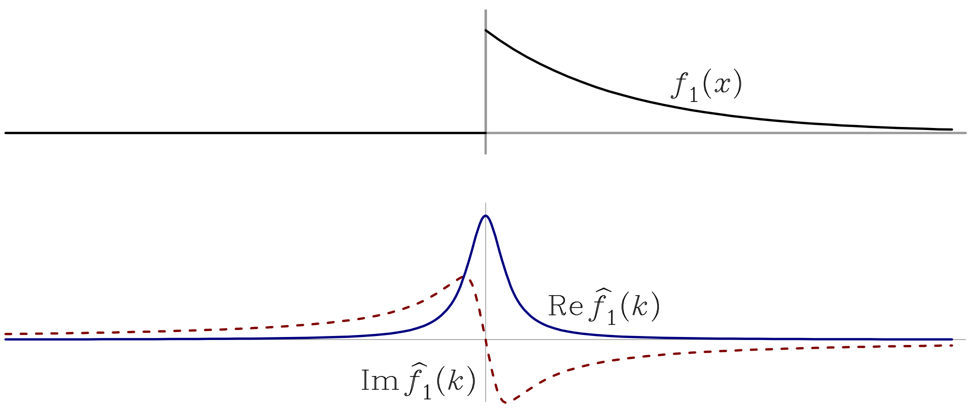
Suppose we wanted to make an even function from (3.1), that is take its even part. From the property shown in Exercise 3 in section 1 for have a general result:

$$\mathcal{F}[f(t) + f(-t)] = \hat{f}(\nu) + \hat{f}(\nu)^* = 2\,\mathrm{Re}\,\hat{f}(\nu)\,. \tag{3.6}$$

It follows (almost immediately) from (3.6) and (3.1) that if:

$$f_2(t) = e^{-a|t|} \tag{3.7}$$

Figure 3a: The FT of the exponential transient in (3.1).



$$\hat{f}_2(\nu) = \frac{2a}{a^2 + 4\pi^2\nu^2} \,. \tag{3.8}$$

Can you see the slight hitch in this argument that needs to be fixed?

The property that $\mathcal{F}^2[f] = f(-t)$ gets us a free FT for every one we calculate. Suppose I want the FT of the function in (3.8); this function is already an FT itself, so taking its FT lands up with the original, with reversed argument. So, after applying a little scaling we find

$$f_3(t) = \frac{1}{b^2 + t^2} \tag{3.9}$$

$$\hat{f}_3(\nu) = \frac{\pi}{b}\, \mathrm{e}^{-2\pi b|\nu|} \,. \tag{3.10}$$

Notice this FT would be hard to compute directly, unless one is familiar with contour integration.

Next we consider a function that is zero outside an interval; the following function is often called a **box car** function because its graph resembles the profile of a piece of railway rolling stock:

$$f_4(t) = \text{box}\,(t) = \begin{cases} 1, & |t| \le ½ \\ 0, & |t| > ½ \,. \end{cases} \tag{3.11}$$

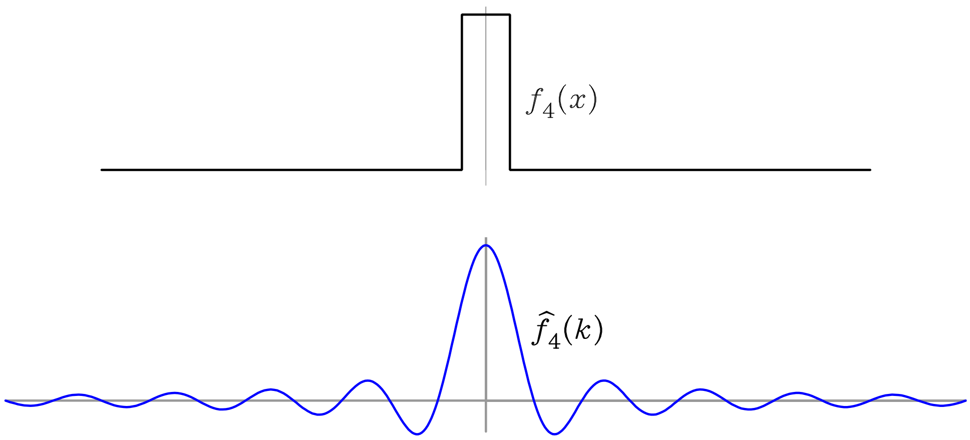
The integral is easy because contributions vanish outside the interval (–½, ½):

$$\hat{f}_4(\nu) = \int_{-½}^{½} \mathrm{e}^{-2\pi \mathrm{i}\nu t}\, dt = \left[\frac{\mathrm{e}^{-2\pi\mathrm{i}\nu t}}{-2\pi\mathrm{i}\nu} \right]_{t=-½}^{t=½} \tag{3.12}$$

$$= \frac{\mathrm{e}^{-\pi\mathrm{i}\nu} - \mathrm{e}^{\pi\mathrm{i}\nu}}{-2\pi\mathrm{i}\nu} = \frac{\sin \pi\nu}{\pi\nu} \,. \tag{3.13}$$

The function $\sin(\pi t)/\pi t$ is often called the **sinc function**. From the same argument

Figure 3b: The FT of the box car is a sinc.



as given before, we see that

$$\mathcal{F}[\text{sinc}\ (t)] = \text{box}\ (\nu)\,. \tag{3.14}$$

Thus the sinc function has no frequency cntent outside the interval $)-½, ½)$. Functions like this are called **band limited** functions.

The various functions encountered in this section are quite unlike their FTs. Now we come to the most famous function that is its own FT, the Gaussian hump. Our proof of this is different from most of the ones you will see in the books, because I want to show you the use of two of the important properties of the FT. To avoid confusing subscripts we write

$$g(t) = \mathrm{e}^{-\pi t^2}\,. \tag{3.15}$$

Then, differentiating we find a simple differential equation for g:

$$\frac{dg}{dt} = -\,2\pi t\; g \tag{3.16}$$

with the initial condition $g(0) = 1$. Take the FT on both sizes of (3.16) and use the differential property and the product property:

$$2\pi \mathrm{i}\nu\; \hat{g}(\nu) = -\,2\pi\, \frac{-1}{2\pi \mathrm{i}}\, \frac{d\hat{g}}{d\nu} \tag{3.17}$$

which upon rearrangement is:

$$\frac{d\hat{g}}{d\nu} = -\,2\pi\nu\; \hat{g} \tag{3.18}$$

which is identical with (3.16). This tells us that $\hat{g}(\nu) = \beta\, g(\nu)$, where $\beta = \hat{g}(0)$, an unknown constant. To find β we apply the FT twice:

$$g(t) = g(-t) = \mathcal{F}^{\,2}[g(t)] = \beta^2\, g(t) \tag{3.19}$$

and so $\beta = \pm 1$. From the definition of the FT, $\hat{g}(0)$ is the area under the (bell) curve $g(t)$ and that is clearly positive, so $\beta = 1$. Thus g is exactly its own FT:

$$\hat{g} = g \quad \text{or} \quad \mathcal{F}\left[\mathrm{e}^{-\pi t^2}\right] = \mathrm{e}^{-\pi \nu^2}\,. \tag{3.20}$$

Perhaps surprisingly, there are infinitely many other functions that are their own FTs. Perhaps you can think of some others.

Exercises

1. We can use our examples to perform some interesting integrals. Observe from (1.1) and (1.2) that

$$\hat{f}(0) = \int_{-\infty}^{\infty} f(t)\, dt, \quad \text{and} \quad f(0) = \int_{-\infty}^{\infty} \hat{f}(\nu)\, d\nu\,.$$

(a) Use (3.13) and other properties of the FT to show that

$$\int_{-\infty}^{\infty} \frac{\sin t}{t}\, dt = \int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2}\, dt = \pi \,.$$

(b) Similarly establish the following famous result from (3. 19):

$$\int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi}\,.$$

(c) Next, show by the change of variables $2\pi\nu = \tan\theta$ that for the function $\hat{f}_1$ defined in (3.8):

$$\int_{-\infty}^{\infty} \hat{f}_1(\nu)\, d\nu = \int_{-\infty}^{\infty} \frac{d\nu}{1 + 4\pi^2\nu^2} = ½\,.$$

But according to (3.1) $f_1(0) = 1$. What has happened, and can we trust the other integrals evaluated in this exercise?

2. Use the stretching scaling relationships to obtain the result given in (1.5). Find the FT of $f(t) = \mathrm{sinc}\,(a[t - t_0])$ and, using MATLAB or otherwise, plot f and $\hat{f}$ for $a = 1$ and $t_0 = 20$.

3. Find the FT of $f(t) = |t|^{-½}$ and of $g(t) = |t|^{½}/t$ (they are different functions!). Hence evaluate the integrals

$$I_1 = \int_0^{\infty} \frac{\cos t}{\sqrt{t}}\, dt \quad \text{and} \quad I_2 = \int_0^{\infty} \frac{\sin t}{\sqrt{t}}\, dt\,.$$

Hint: Find $\hat{f}(\gamma\nu)$ for a constant γ by (1.1) and (2.1) and compare; then choose γ judiciously.

A Very Short Table of Fourier Transforms

$f(t)$	$\hat{f}(\nu)$
$e^{-at}H(t)$	$\dfrac{1}{a + 2\pi i\nu}$
$e^{-a\lvert t\rvert}$	$\dfrac{2a}{a^2 + 4\pi^2\nu^2}$
$\dfrac{1}{b^2 + t^2}$	$\dfrac{\pi}{b}\, e^{-2\pi b\lvert\nu\rvert}$
$\mathrm{box}\,(t)$	$\mathrm{sinc}\,(\nu)$
$\mathrm{sinc}\,(t)$	$\mathrm{box}\,(\nu)$
$e^{-\pi t^2}$	$e^{-\pi\nu^2}$

The constants a and b are always real and positive. Also the **Heaviside step function** is defined by

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \ge 0 \end{cases}$$

4. Convolution

For this segment let us appeal to a physical model of our function as a signal from an instrument. Imagine a seismometer, for example, whose **impulse response** is $u(t)$, which means that an input pulse of unit height and short duration dt generates output of the form $u(t)\,dt$. The function given by f_1 in (3.1) could be a suitable candidate. Now consider what happens when a continuous signal $s(t)$ in the ground excites the seismometer. The output at a particular time t_0 is the sum (really the integral) of all the contributions from the impulse responses at all previous times. We can write the result

$$r(t_0) = \int_{-\infty}^{t_0} dt\ s(t)\,u(t_0 - t) \tag{4.1}$$

which you will see is the sum of all impulses, weighted by their amplitudes, up to the time t_0. If, as is the case with f_1 the system has no response before $t = 0$ (it is a **causal system**) then we can replace the upper limit on the integral to obtain:

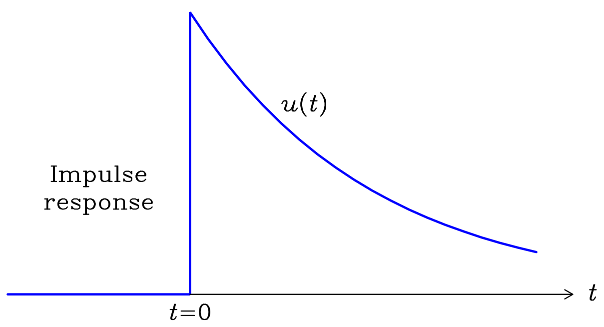
$$r(t_0) = \int_{-\infty}^{\infty} dt\ s(t)\,u(t_0 - t)\,. \tag{4.3}$$

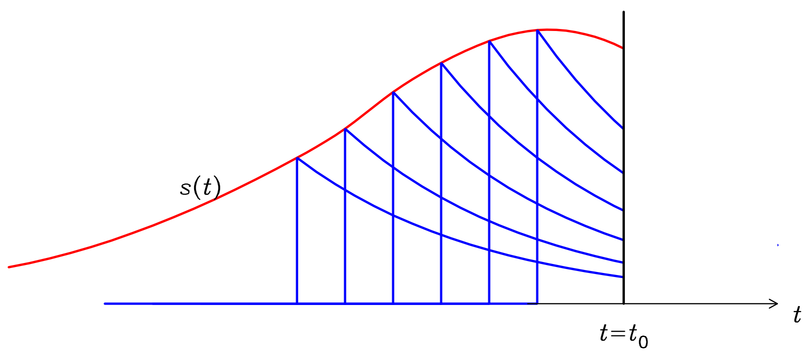
This integral is called a **convolution** and is often written

$$r = s * u \tag{4.4}$$

because the operation of convolution has many of the properties of a multiplication. It will be obvious that, like ordinary multiplication, it is distributive with addition: $s*(u+v) = s*u + s*v$. Less obvious are the properties of commutation: $s*u = u*s$ and associative property: $s*(u*v) = (s*u)*v = s*u*v$.

Figure 4a: Building the output from the impulse response.





We should note at this point that, while causal response functions are the only kind that make sense when u is the impulse response of a physical system in time, when the independent variable represents space, that will not usually be the case. For example, if we were calculating the magnetic field from a thin layer of magnetized material, in a two-dimensional approximation, then $u(t)$ would be the field of a single line of dipoles perpendicular to the observation plane, and the function u then extends indefinitely in the positive and negative t directions. We shall see that convolution is easily generalized into more than one spatial dimension, where it usually represents the result of integration over a set of point sources, like point masses, or point dipoles.

Next we state the **Convolution Theorem**: it states that

$$\mathcal{F}[f * g] = \hat{f}\,\hat{g} \tag{4.5}$$

In words: the FT of convolution is just the product of the FTs of the two functions. The convolution theorem is fairly straightforward to prove: it will be a homework problem. One reason the convolution theorem is important in modern times arises from the fact that we can compute approximate FTs very quickly with an algorithm called the **Fast Fourier Transform**, or FFT. Convolutions done numerically by approximating the integral in (4.1) will require a number of arithmetic operations involving the product of three terms: $N_u N_s N_r$, where N_u is the number of terms in the series for the signal s, and similarly for the other factors. When all three numbers are large, this can be a lot of computing, and this will happen regularly in the 2- and 3-dimensional versions of convolution. Then it will be the case that it is numerically more efficient to perform the arithmetic using the following identity:

$$u * s = \mathcal{F}^{-1}[\hat{u} \cdot \hat{s}] \tag{4.6}$$

because the numerical work to perform an FT grows like $N \log N$; three numerical FTs will be much faster than one convolution for large N. In fact MATLAB automatically uses FFTs to do long convolutions without telling you. We will look at numerical approximation of FTs in the next section.

Let us return to our seismometer. The **impulse response** $u(t)$ of a seismometer is defined as the output when the input is a spike of infinitesimal duration. Then the output $r(t)$ caused by an extended input $s(t)$ is given by the convolution:

$$r(t) = s * u = \int_{-\infty}^{\infty} s(p)\, u(t-p)\; dp \tag{4.7}$$

The **frequency response** of the seismometer is the complex number that gives the amplitude and phase of the sine-wave output, when a sinusoidal impulse is the input. The idea behind the frequency response is that linear systems can usually be viewed as **filters**: a particular band of frequencies may be attenuated, while another band is accentuated. There is a vast literature on designing filters in engineering to remove parts of the signal, but let pass unaffected signals in the so-called pass band. Natural systems have their own frequency responses. We can calculate the frequency response of the seismometer easily from the impulse response, but setting the input in (4.7) to be the complex exponential with frequency ν:

$$s(t) = e^{2\pi i\nu t} = \cos 2\pi\nu t + i \sin 2\pi\nu t \tag{4.8}$$

which is an elegant way of presenting both a cosine and sine signal of unit amplitude to the instrument. Recall $s * u = u * s$, and so

$$r(t) = \int_{-\infty}^{\infty} e^{2\pi i(t-p)\nu} u(p)\, dp = e^{2\pi i\nu t} \int_{-\infty}^{\infty} e^{-2\pi i\nu p} u(p)\, dp \tag{4.9}$$

$$= e^{2\pi i\nu t}\, \hat{u}(\nu)\,. \tag{4.10}$$

This equation tells us immediately that *the frequency response is just the Fourier transform of the impulse response.*

To illustrate what this means in physical terms let us break things down into real and imaginary parts as in (4.8). We write the transform $\hat{u}$ as

$$\hat{u}(\nu) = |\, \hat{u} \,|\, e^{i\phi(\nu)} \tag{4.11}$$

Then from (4.10) we see

$$r(t) = |\, \hat{u}(\nu) \,|\, e^{i\phi(\nu)}\, e^{2\pi i\nu t} = |\, \hat{u}(\nu) \,|\, e^{i\phi(\nu)+2\pi i\nu t} \tag{4.12}$$

$$= |\, \hat{u}(\nu) \,|\, [\cos(\phi(\nu)+2\pi\nu t) + i \sin(\phi(\nu)+2\pi\nu t)]\,. \tag{4.13}$$

Comparing the real parts of (4.8) and (4.13) we see that (a) the input is a cosine, and so is the output of the same frequency; (b) the output does not have unit amplitude: its amplitude is $|\, \hat{u}(\nu) \,|$, the magnitude of the frequency response; (c) a phase shift has been introduced by the instrument, which is given by $\phi(\nu)$. For real instruments the phase is negative because the system delays the input by a certain amount, since it cannot predict the future. Thus the magnitude and phase of the frequency response tells us the amplitude and phase shift imparted to the signal by the seismometer. Now look at the imaginary parts of (4.10) and (4.13). Exactly the same thing happens: the same amplitude and phase shift occurs for a sine wave as a cosine.

Next we will calculate the frequency response of a model seismometer. If the vertical displacement of the seismometer beam relative to a point fixed in the ground is given by $z(t)$, and the ground acceleration is $a(t)$, the differential equation of motion, from Newton's second law is

$$\frac{d^2 z}{dt^2} + \frac{\omega_0}{Q}\frac{dz}{dt} + {\omega_0}^2 z(t) = a(t) \tag{4.14}$$

where ω_0 is the radian frequency of the seismometer resonance, and Q is the so-called quality factor, an inverse measure of the damping in the system: the amplitude of oscillation dies away by a factor e^{-1} in Q cycles, so high Q means low damping. To find the frequency response we can take the FT of both sides of (4.14), and use the properties. In particular, recall that differentiating brings out a multiplicative factor of $2\pi i\nu$, so doing that twice squares the factor:

$$-4\pi^2\nu^2\, \hat{z}(\nu) + \frac{2\pi i\nu\omega_0}{Q}\, \hat{z}(\nu) + {\omega_0}^2\, \hat{z}(\nu) = \hat{a}(\nu) \tag{4.15}$$

Solving for $\hat{z}$ we find

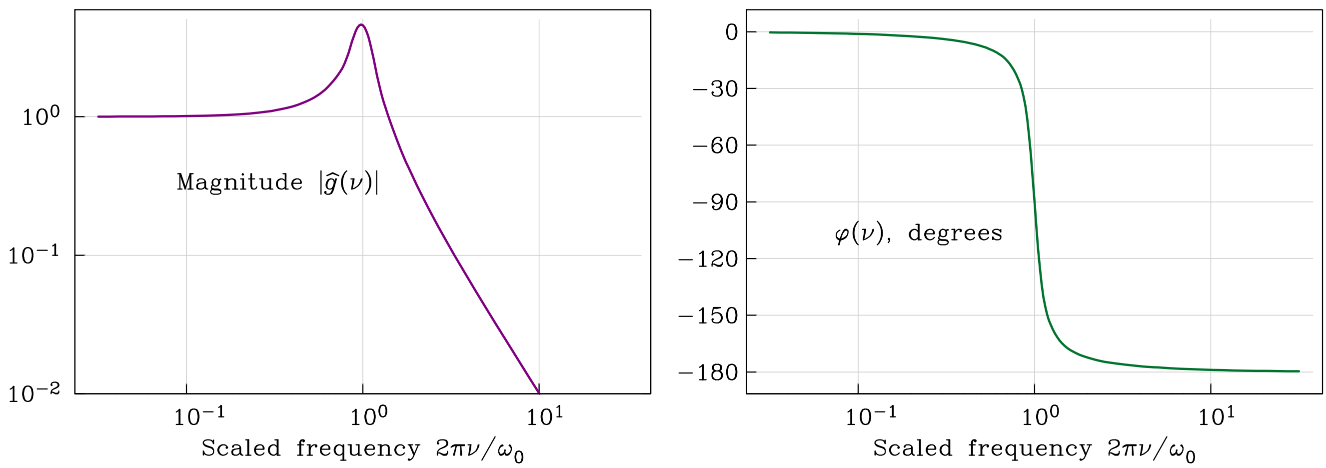
$$\hat{z}(\nu) = \frac{\hat{a}(\nu)}{{\omega_0}^2 - 4\pi^2\nu^2 + 2\pi i \nu \omega_0/Q} = \hat{a}(\nu)\,\hat{g}(\nu)\,. \tag{4.16}$$

According to the Convolution Theorem, if we took the inverse FT of (4.16), we would get the output signal as the convolution of the ground acceleration and the function $\hat{g}$. But from (4.7), this means that $g(t)$ is the impulse response, and $\hat{g}(\nu)$ is the frequency response. So we have found the frequency response directly from the differential equation, without having to solve it! This shows an example of how the FT is so useful. As I mentioned earlier solutions to many differential equations look like convolutions.

The graph below shows the magnitude and phase responses of the model seismometer with a Q value of 5. Notice that below a frequency of about 0.3 dimensionless units the seismometer has a flat frequency response, meaning that the ground acceleration is faithfully reflected in the beam position, but higher frequency signals are increasing cut out. Observe too how the phase response is zero in the lower band, but then turns to 180° in the upper band, so that the bean is moving in the opposite direction to the ground acceleration there.

An important question arises: Given the output of an instrument like a seismometer with known impulse response, can we recover the original input signal $s(t)$ from the measured output $r(t)$ in (4.7)? This is the question of **deconvolution**. At first glance the answer seems easy using the Convolution Theorem: since $\hat{r} = \hat{u}\,\hat{s}$, and $\hat{u}$ is known, we just divide by $\hat{u}$ and then take the inverse FT: $s = \mathcal{F}^{-1}[\hat{r}/\hat{u}]$. Unfortunately in most practical systems (like the seismometer response below) high-frequencies are strongly attenuation so that $\hat{h}$ gets very small, which means the recovered signal is obtained by amplifying small values. These small values are often submerged by noise in the system, and so this deconvolution succeeds only in amplifying noise, not true signal. Deconvolution is therefore an unstable procedure and straight division of the FTs hardly ever works; the process must be **regularized**, a topic we will meet in inverse theory.

Figure 4b: Seismometer frequency response with $Q = 5$.



Exercises

1(a) Use equation (2.7) and other properties in Section 2 to prove the Convolution Theorem.

(b) Use the Convolution Theorem to find the FT of $\mathrm{sinc}^2(x)$. Plot the FT.