Four Energy Problems

In this note I give the math behind various assertions made in my January and February columns in the *Freewheeler*, the North County Cycle Club's newsletter.

Problem I

The simplest problem concerns a cyclist heading out and back on a flat straight (and very dull) ride on two days: the first is calm with no wind; on the second day a steady wind blows exactly along the direction of the route. In both cases he rides at the same speed v. We want to show more energy is used when the wind blows: even though the rider is helped by a tail wind on the way out, this doesn't make up for the extra work on the way back, because of the square law for air resistance.

On the calm day the work done, which equals the energy expended by the rider, is given by force times distance:

$$W_1 = Cv^2 \times 2L = 2LCv^2$$

where C is a constant (which depends on he shape of the cyclist among other things) and L is the one-way distance. On the windy day if the wind speed w is less than v the force opposing the cyclist when he or she rides down wind is less, namely $C(v-w)^2$, because the air-speed relative to the rider is $v-w$. And riding into the wind the force is of course more, since the headwind adds to his/her speed: $C(v+w)^2$. The energy required is the sum of the out and back contributions:

$$W_2 = C(v-w)^2 \times L + C(v+w)^2 \times L = LC\left[(v-w)^2 + (v+w)^2\right].$$

Now we expand the squares in the brackets in the usual way:

$$W_2 = LC[v^2 - 2vw + w^2 + v^2 + 2vw + w^2]$$

$$= 2LC[v^2 + w^2].$$

We can see at once that this quantity is greater than W_1 because $w^2 > 0$ as long as w is not zero. Thus in this highly simplified scenario it is always more work to ride in the wind.

When the wind blows from the side, as it always will at some point because real routes are not perfectly straight, we need to put in the directions which complicates things, and I will not go into the analysis here: it still comes out a net loss, even if the wind is exactly perpendicular to he route. Another simplification comes from the approximation of using the square law, something that shows up if we say that $w \geq v$. If the tail wind speed exactly equals the rider's speed he or she may still need to pedal and do work to overcome other frictional forces, such as those with road and in the hubs. But the conclusion, borne out by experience, holds up: it's always more tiring to ride a round-trip on a windy day. This is a little harder than the first problem.

Problem II

Two routes of equal length join point A to point B; one is entirely flat, the other has a uniform grade to the midpoint, then drops uniformly to B. On a windless day a cyclist does the loop, taking the flat route out and returning on the hilly road, coasting downhill. The rider completes each half in equal times. Does the hilly part of the course from B to A always use more energy?

Solution

First set up definitions of some variables for the calculation. The drag force follows Newton's law: $D = Cv^2$. The ride out is at velocity v_1, the uphill ride at v_2 and the downhill part at v_3 and the grade angle be θ and the distance from A to B be L.

In all these calculations the energy expended equals the work done: distance times opposing force. The energy expended riding on the flat is just the distance traveled times the drag force:

$$E_F = LCv_1^2.$$

Going uphill the work done lifting the cyclist must be included. Remember the distance is halved, so the energy used ascending is

$$E_H = \tfrac{1}{2}L(mg \sin\theta + Cv_2^2)$$

where mg is the weight of the rider plus the bike. I have taken the constant C to be the same here, since pedaling uphill and on the flat are similar enough (no standing). No energy is expended by the cyclist on the descent; then the drag and the force of gravity balance:

$$cv_3^2 = mg \sin\theta.$$

Here the drag constant is different; for a rider in a crouch $c<C$. Since the times on the outward and homebound legs are equal

$$\frac{L}{v_1} = \frac{\tfrac{1}{2}L}{v_2} + \frac{\tfrac{1}{2}L}{v_3}.$$

At this point we have all the relationships we need to solve the problem.

Eliminate the gravitational force $mg \sin\theta$ from the second energy equation by using the downhill force balance:

$$E_H = \tfrac{1}{2}L(Cv_2^2 + cv_3^2).$$

We are interested in the relationship between E_H and E_F; therefore calculate their ratio:

$$\frac{E_H}{E_F} = \frac{\tfrac{1}{2}L(Cv_2^2 + cv_3^2)}{LCv_1^2} = \frac{v_2^2 + rv_3^2}{2v_1^2}$$

where $r = c/C$. Next, eliminate v_1 using the equality of time out and back:

$$\frac{E_H}{E_F} = \frac{(v_2^2 + rv_3^2)}{8}\left(\frac{1}{v_2} + \frac{1}{v_3}\right)^2 = \frac{1}{8}\left(1 + r\frac{v_3^2}{v_2^2}\right)\left(1 + \frac{v_2}{v_3}\right)^2 .$$

This equation shows the ratio of energy expenditures depends only on the ratio v_2/v_3 and on r the ratio of drag coefficients for aero and normal riding positions. When $v_2 = v_3$ the ratio is $\frac{1}{2}(1 + r)$, which is less than one if $r < 1$, as I said in my column this month. So for example, if you can reduce your air resistance by 30% (so that $r = 0.7$), you can potentially use 15% less energy on the hilly leg of the the ride. Is this the best that can be done, or can the energy-use ratio be less than this?

That question can be answered by elementary calculus. Write $v_3/v_2 = u$ and E_H/E_F as Y. Then after a bit of algebra
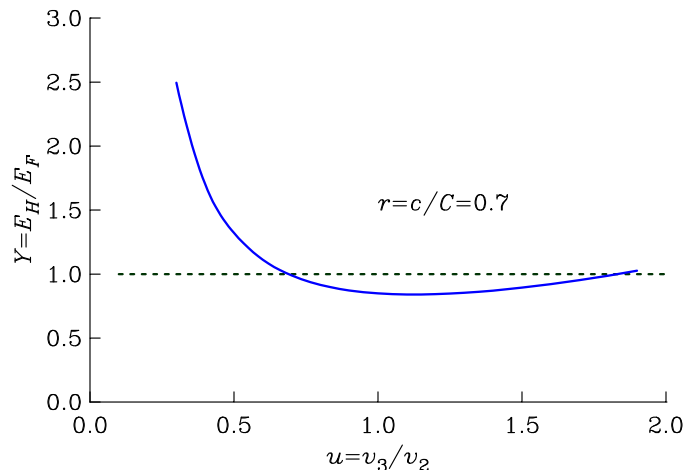
$$8Y = 1 + r + \frac{2}{u} + \frac{1}{u^2} + 2ru + ru^2$$

$$8\frac{dY}{du} = 2ru + 2r - \frac{2}{u^2} - \frac{2}{u^3} = 2(u + 1)\left(r - \frac{1}{u^3}\right).$$

The minimum Y arises when $dY/du = 0$, and for positive u this is evidently when $u = r^{-1/3}$. Plugging this into the equation for Y gives

$$Y_{\min} = \frac{1}{8}(1 + r^{1/3})^3 .$$

Returning to the example with $r = 0.7$ we find that $Y_{\min} = 0.841$ or 15.9% reduction. The improvement over 15% from the equal-velocity strategy with $v_3/v_2 = u = 1$ is insignificant.

Problem III

The start and end of the ride are at the same level, with a constant grade to the crest and constant grade down, but the crest is not necessarily at the midpoint. A cyclist must ride over this hill without coasting and with minimum energy use at a specified average speed. Show that maintaining the same speed up the grade and down uses the least energy.

Solution

We will use some advanced calculus for this variant, which shortens the solution considerably. This solution is for experts only. Now we let the average speed be v_0, the speed uphill be v_1, and that downhill v_2. The length of the uphill portion is L_1 and downhill L_2; the angle of the grades are θ_1 and θ_2. The will assign the same drag coefficient up and down the grade (though it would be somewhat smaller downhill). The total work done is the sum of the up and down portions:

$$E = L_1(cv_1^2 + mg \sin \theta_1) + L_2(cv_2^2 - mg \sin \theta_2).$$

On the downhill leg gravity reduces the force, hence the negative sign. Also we assume that the cyclist is pedaling, so that $cv_2^2 \geq mg \sin \theta_2$. Notice that $L_1 \sin \theta_1 = L_2 \sin \theta_2 = h$, the height of the hill, and so the equation simplifies to

$$E = c(L_1 v_1^2 + L_2 v_2^2).$$

The average speed must be v_0; hence

$$\frac{L_1 + L_2}{v_0} = \frac{L_1}{v_1} + \frac{L_1}{v_2}.$$

To minimize E subject to this constraint, we introduce the *Lagrange multiplier* λ, and the unconstrained function F:

$$F(v_1, v_2, \lambda) = E(v_1, v_2) - \lambda \left(\frac{L_1 + L_2}{v_0} - \frac{L_1}{v_1} - \frac{L_1}{v_2} \right).$$

The minimum energy point occurs when

$$\frac{\partial F}{\partial v_1} = 0; \quad \frac{\partial F}{\partial v_2} = 0; \quad \frac{\partial F}{\partial \lambda} = 0.$$

The first two equations yield

$$2cL_1 v_1 - \frac{\lambda L_1}{v_1^2} = 0 \ \text{ or } \ v_1^3 = \frac{\lambda}{2c}$$

$$2cL_2 v_2 - \frac{\lambda L_2}{v_2^2} = 0 \ \text{ or } \ v_2^3 = \frac{\lambda}{2c}.$$

Comparing these two equations shows without further algebra that $v_1 = v_2$ as required.

Proving this stationary point of the system yields a true minimum is most easily done by appealing to the convexity of the energy function, and writing the speed constraint as an inequality (the average speed must be at least v_0), which leads to a convex constraint set.

Problem IV

Returning to Problem I, we ask whether it always takes more work when a steady wind is blowing to ride a given route out and back at a constant speed than when the air is still. And that route is allowed to be of any shape. The answer is yes, provided the wind is constant in speed and direction, more energy is required of the cyclist when it is blowing.

Solution

First we analyze a straight section. The rider rides along the road with vector velocity $\mathbf{v}$ and the wind blows with constant velocity $\mathbf{w}$, both of these being constant in time and nonvanishing. The power required to maintain that speed is

$$P = -\,\mathbf{v} \cdot \mathbf{D}(\mathbf{v}, \mathbf{w})$$

where $\mathbf{D}$ is the drag, given by Newton's law

$$\mathbf{D}(\mathbf{v}, \mathbf{w}) = c\,|\,\mathbf{w} - \mathbf{v}\,|\,(\mathbf{w} - \mathbf{v})\,.$$

This result holds only if the power required is positive, that is, the rider does work against the air, which means he or she is riding faster than the wind speed. We will proceed on this assumption for the moment. Since the time, T, taken in either direction is the same by hypothesis, the energy expended is PT. We sum the work done in both directions:

$$W = -\,T[\mathbf{v} \cdot \mathbf{D}(\mathbf{v}, \mathbf{w}) + (-\mathbf{v}) \cdot \mathbf{D}(-\mathbf{v}, \mathbf{w})]\,.$$

Our aim is to show this must be greater than $2cT\,|\,\mathbf{v}\,|^3$, the work done in still air.

Substituting the expression for the drag into the equation for W and reorganizing trivially we get

$$\frac{W}{cT} = |\,\mathbf{v} - \mathbf{w}\,|\,(\mathbf{v} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w}) + |\,\mathbf{v} + \mathbf{w}\,|\,(\mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w})$$

$$= |\,\mathbf{v}\,|^2 (|\,\mathbf{v} + \mathbf{w}\,| + |\,\mathbf{v} - \mathbf{w}\,|) + \mathbf{v} \cdot \mathbf{w}(|\,\mathbf{v} + \mathbf{w}\,| - |\,\mathbf{v} - \mathbf{w}\,|)\,.$$

In the first term, we can use the triangle inequality for vectors that states for any two vector

$$|\,\mathbf{a}\,| + |\,\mathbf{b}\,| \geq |\,\mathbf{a} + \mathbf{b}\,|\,.$$

Equality holds only if $\mathbf{a} = 0$ or $\mathbf{b} = \gamma\mathbf{a}$ with $\gamma \geq 0$. Hence

$$|\,\mathbf{v} + \mathbf{w}\,| + |\,\mathbf{v} - \mathbf{w}\,| \geq 2\,|\,\mathbf{v}\,|\,.$$

And therefore

$$\frac{W}{cT} \geq 2\,|\,\mathbf{v}\,|^3 + \mathbf{v} \cdot \mathbf{w}(|\,\mathbf{v} + \mathbf{w}\,| - |\,\mathbf{v} - \mathbf{w}\,|)\,.$$

The second term here can be rewritten

$$\mathbf{v} \cdot \mathbf{w}(|\mathbf{v}+\mathbf{w}| - |\mathbf{v}-\mathbf{w}|) = \mathbf{v} \cdot \mathbf{w} \frac{(|\mathbf{v}+\mathbf{w}| - |\mathbf{v}-\mathbf{w}|)(|\mathbf{v}+\mathbf{w}| + |\mathbf{v}-\mathbf{w}|)}{|\mathbf{v}+\mathbf{w}| + |\mathbf{v}-\mathbf{w}|}$$

$$= \mathbf{v} \cdot \mathbf{w} \frac{|\mathbf{v}+\mathbf{w}|^2 - |\mathbf{v}-\mathbf{w}|^2}{|\mathbf{v}+\mathbf{w}| + |\mathbf{v}-\mathbf{w}|} = \frac{4(\mathbf{v} \cdot \mathbf{w})^2}{|\mathbf{v}+\mathbf{w}| + |\mathbf{v}-\mathbf{w}|}.$$

This expression is positive unless one of the vectors is zero or they are orthogonal. Thus we have shown that

$$\frac{W}{cT} > 2|\mathbf{v}|^3.$$

The possibility of equality can be eliminated since neither $\mathbf{v}$ or $\mathbf{w}$ vanishes, and the triangle inequality is strict when $\mathbf{v}$ and $\mathbf{w}$ are orthogonal and of positive length.

We return to the question of what happens if the wind speed is greater than the rider's, and the power $P < 0$. While the rider may be able to coast and do zero work, (which is $P = 0$) he or she cannot store up energy from the wind to be used later, which is what $P < 0$ implies. The restriction to nonnegative P only strengthens the inequality already derived.

We have established the result over a straight section with the wind in any direction. For a route of arbitrary shape, we imagine its decomposition into infinitesimally short straight segments, on which the inequality applies uniformly. Therefore it applies to the union of those segments which means the inequality holds for the complex route too.