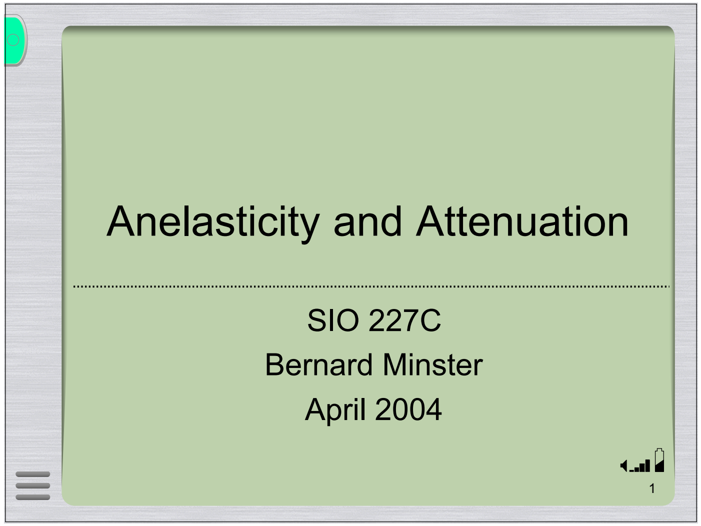
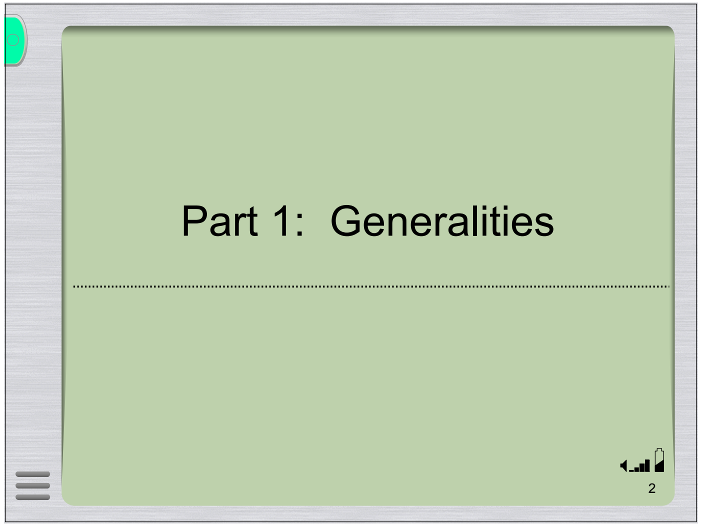
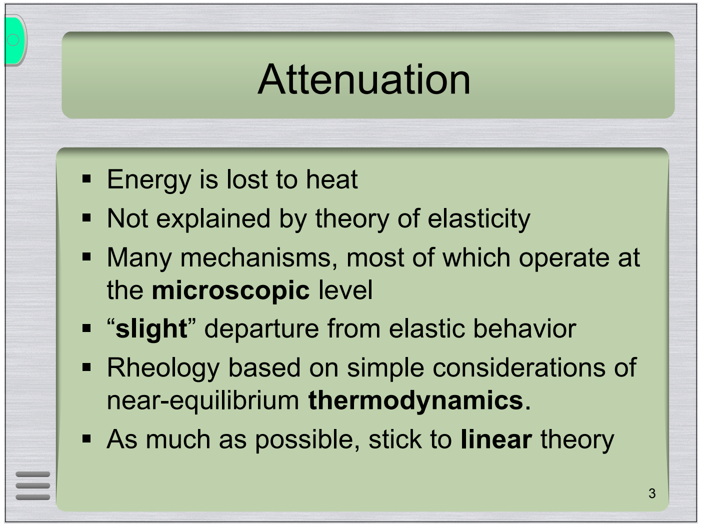


Anelasticity and Attenuation

SIO 227C
Bernard Minster
April 2004

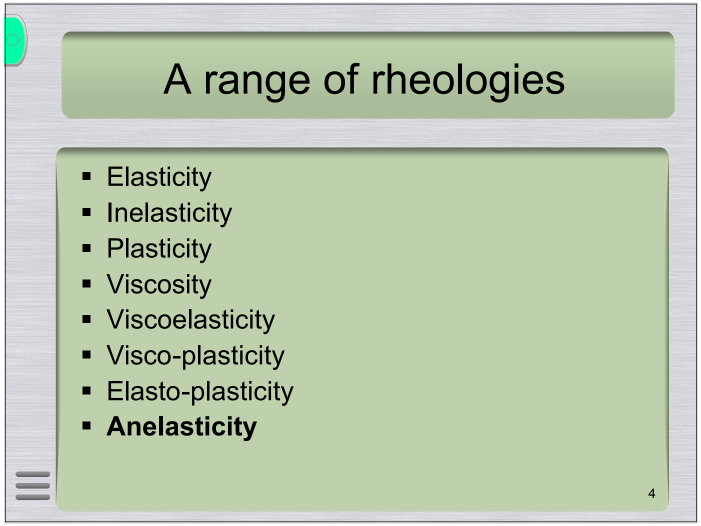


Part 1: Generalities



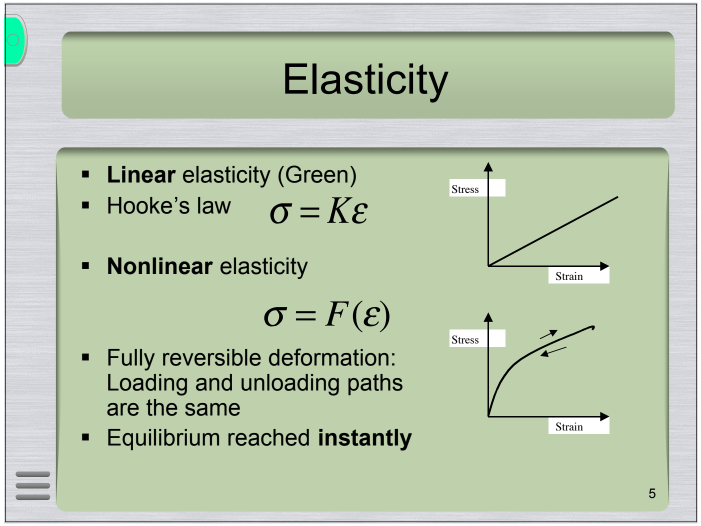
Attenuation

- Energy is lost to heat
- Not explained by theory of elasticity
- Many mechanisms, most of which operate at the **microscopic** level
- “**slight**” departure from elastic behavior
- Rheology based on simple considerations of near-equilibrium **thermodynamics**.
- As much as possible, stick to **linear** theory



A range of rheologies

- Elasticity
- Inelasticity
- Plasticity
- Viscosity
- Viscoelasticity
- Visco-plasticity
- Elasto-plasticity
- **Anelasticity**

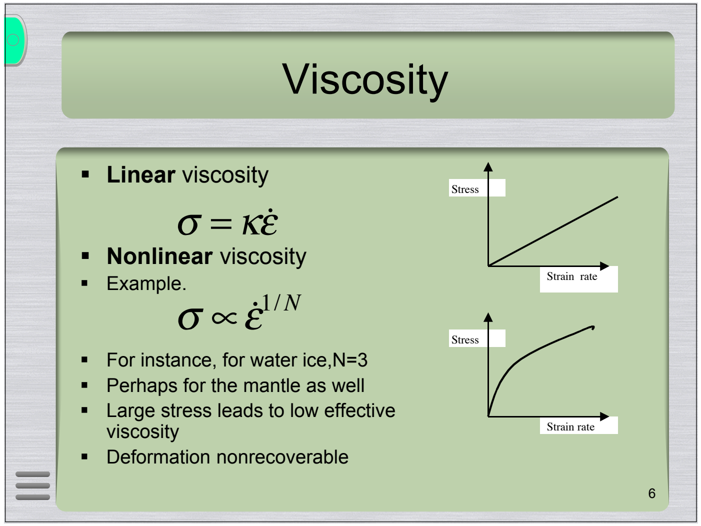


Elasticity

- **Linear** elasticity (Green)
- Hooke's law $\sigma = K\varepsilon$
- **Nonlinear** elasticity

$$\sigma = F(\varepsilon)$$

- Fully reversible deformation: Loading and unloading paths are the same
- Equilibrium reached **instantly**



Viscosity

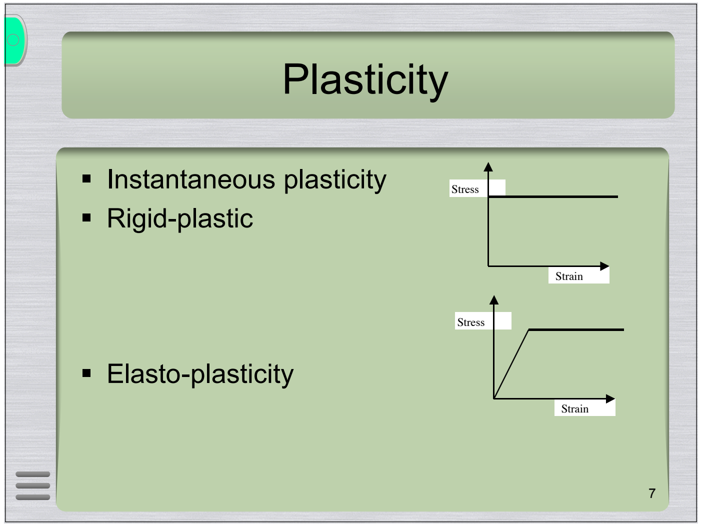
- **Linear** viscosity

$$\sigma = \kappa\dot{\varepsilon}$$

- **Nonlinear** viscosity
- Example.

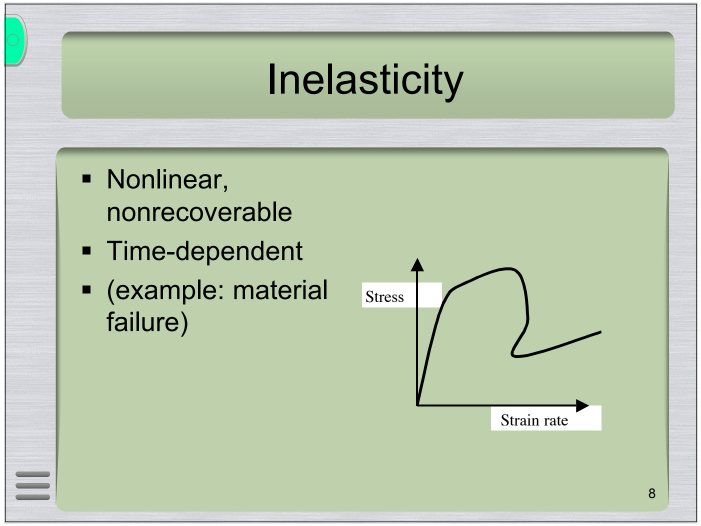
$$\sigma \propto \dot{\varepsilon}^{1/N}$$

- For instance, for water ice,N=3
- Perhaps for the mantle as well
- Large stress leads to low effective viscosity
- Deformation nonrecoverable



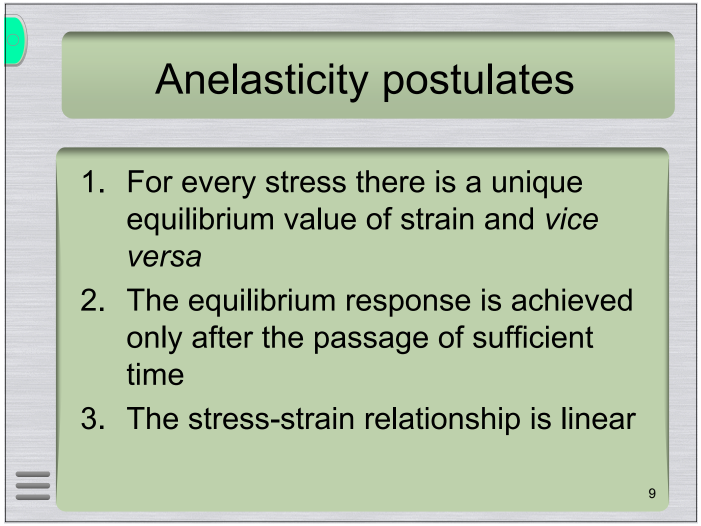
Plasticity

- Instantaneous plasticity
- Rigid-plastic
- Elasto-plasticity



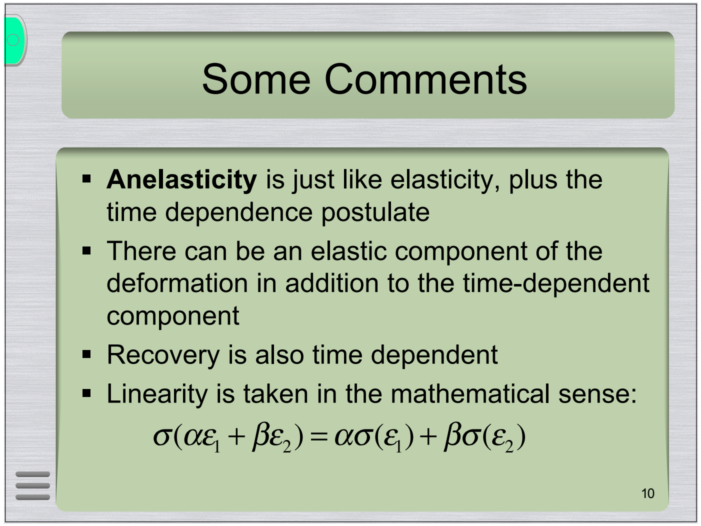
Inelasticity

- Nonlinear, nonrecoverable
- Time-dependent
- (example: material failure)



Anelasticity postulates

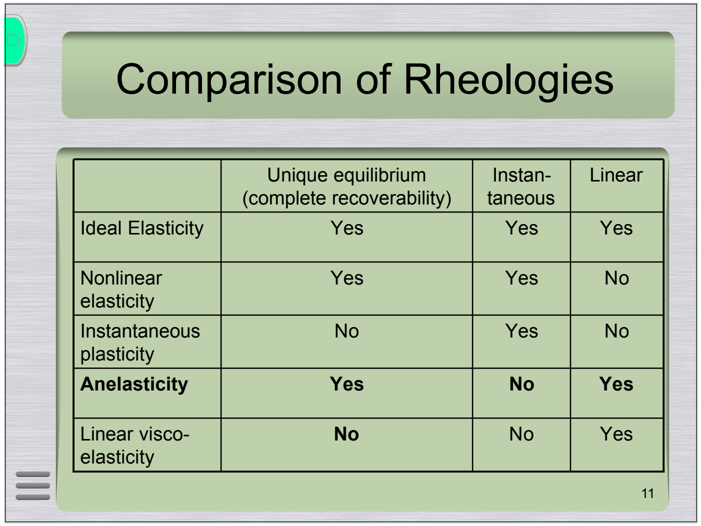
1. For every stress there is a unique equilibrium value of strain and *vice versa*
2. The equilibrium response is achieved only after the passage of sufficient time
3. The stress-strain relationship is linear



Some Comments

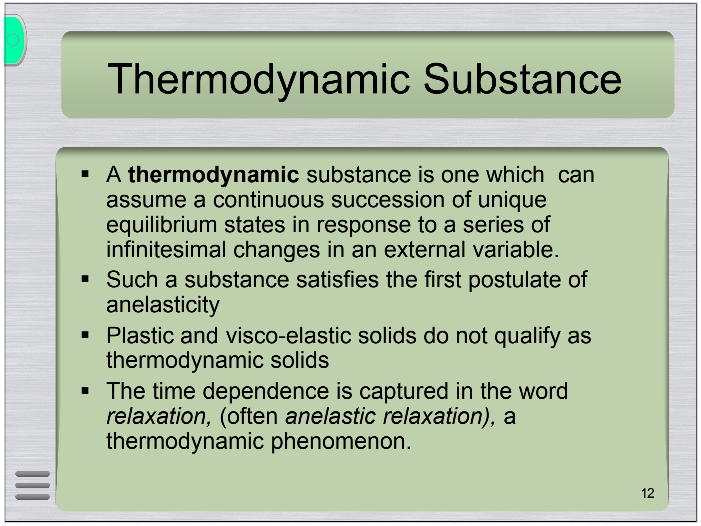
- **Anelasticity** is just like elasticity, plus the time dependence postulate
- There can be an elastic component of the deformation in addition to the time-dependent component
- Recovery is also time dependent
- Linearity is taken in the mathematical sense:

$$\sigma(\alpha\varepsilon_1 + \beta\varepsilon_2) = \alpha\sigma(\varepsilon_1) + \beta\sigma(\varepsilon_2)$$



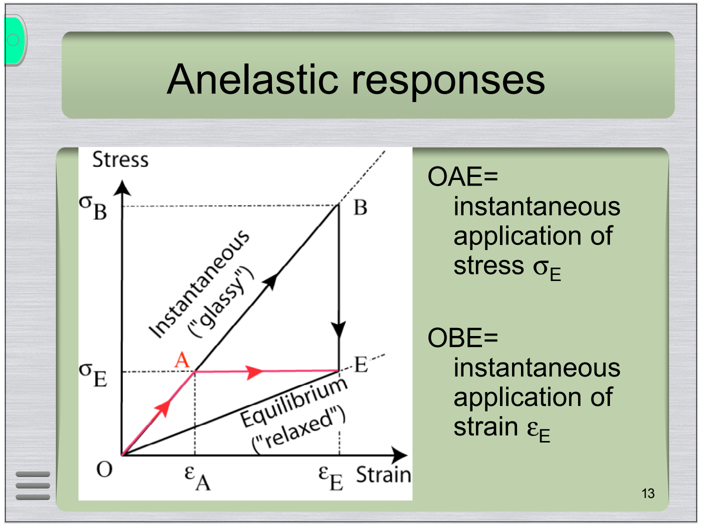
Comparison of Rheologies

	Unique equilibrium (complete recoverability)	Instan-taneous	Linear
Ideal Elasticity	Yes	Yes	Yes
Nonlinear elasticity	Yes	Yes	No
Instantaneous plasticity	No	Yes	No
Anelasticity	**Yes**	**No**	**Yes**
Linear visco-elasticity	**No**	No	Yes



Thermodynamic Substance

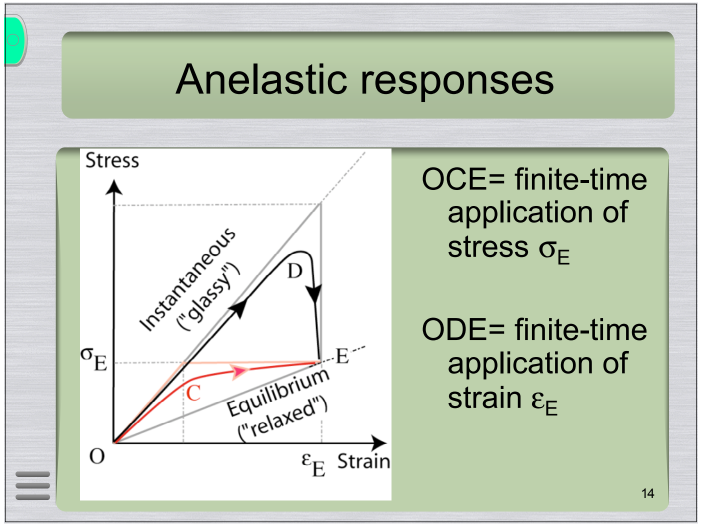
- A **thermodynamic** substance is one which can assume a continuous succession of unique equilibrium states in response to a series of infinitesimal changes in an external variable.
- Such a substance satisfies the first postulate of anelasticity
- Plastic and visco-elastic solids do not qualify as thermodynamic solids
- The time dependence is captured in the word *relaxation,* (often *anelastic relaxation),* a thermodynamic phenomenon.



Anelastic responses

OAE= instantaneous application of stress σ_E

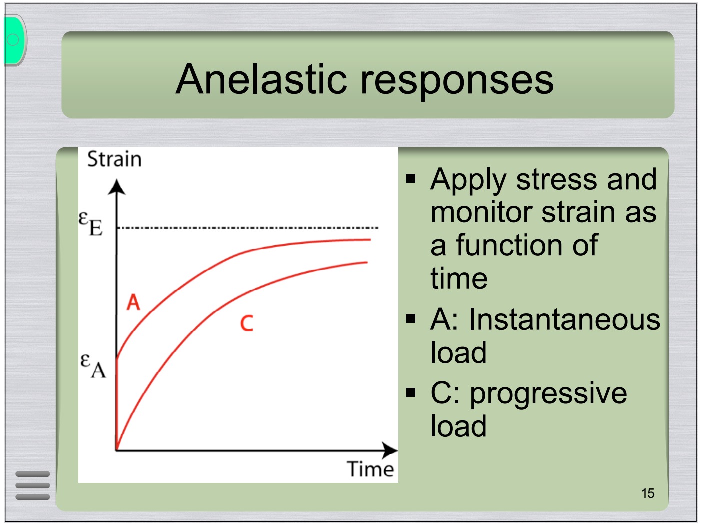
OBE= instantaneous application of strain ε_E



Anelastic responses

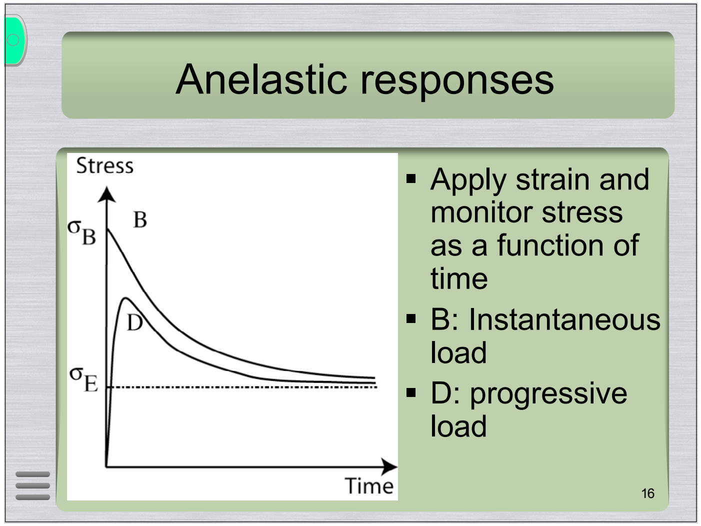
OCE= finite-time application of stress σ_E

ODE= finite-time application of strain ε_E



Anelastic responses

- Apply stress and monitor strain as a function of time
- A: Instantaneous load
- C: progressive load



Anelastic responses

- Apply strain and monitor stress as a function of time
- B: Instantaneous load
- D: progressive load

Creep: definitions

- Creep experiment:

$$\sigma = \begin{cases} 0, & for \quad t<0 \\ \sigma_0 & for \quad t \geq 0 \end{cases}$$

- **Creep function**: Unrelaxed compliance, relaxed Compliance, Compliance relaxation

$$J(t) = \varepsilon(t)/\sigma_0$$
$$J(0) = J_U \quad ; \quad J(\infty) = J_R \quad ; \quad \delta J = J_R - J_U$$

Elastic Aftereffect

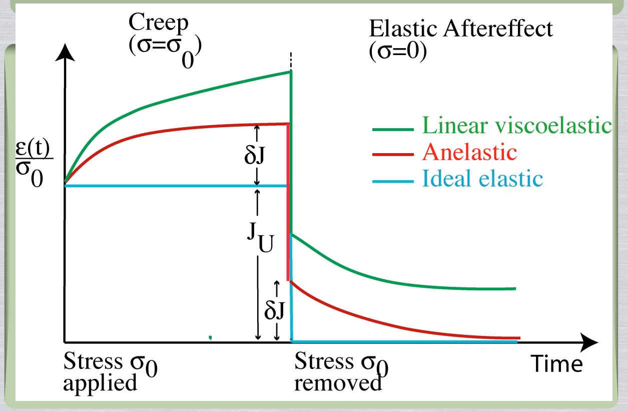
- Recovery experiment

$$\sigma = \begin{cases} 0, & for\ t < -t_1 \\ \sigma_0, & for\ -t_1 \leq t < 0 \\ 0 & for\ t \geq 0 \end{cases}$$

- **Aftereffect function or creep recovery function.** Depends on t_1

$$N_{t_1}(t) = \varepsilon(t)/\sigma_0 \quad for \quad t \geq 0$$

Creep and Recovery



Stress Relaxation

- Stress relaxation experiment

$$\varepsilon = \begin{cases} 0, & for \quad t<0 \\ \varepsilon_0 & for \quad t \geq 0 \end{cases}$$

- **Stress relaxation function**: Unrelaxed modulus, relaxed modulus, modulus defect

$$M(t) = \sigma(t)/\varepsilon_0$$
$$M(0) = M_U \quad ; \quad M(\infty) = M_R \quad ; \quad \delta M = M_U - M_R$$

Relationships

$$M_R = 1/J_R \quad ; \quad M_U = 1/J_U$$

$$J_R > J_U \quad \text{so that} \quad M_R < M_U$$

$$\delta M = \delta J / J_U J_R$$

- **Relaxation Strength** (dimensionless)

$$\Delta = \delta J / J = \delta M / M_R$$

Normalized Creep and Relaxation Functions

- Normalized creep function

$$J(t) = J_U + \delta J \psi(t) = J_U[1 + \Delta\psi(t)]$$

$$\psi(0) = 0 \quad ; \quad \psi(\infty) = 1$$

- Normalized stress relaxation function

$$M(t) = M_R + \delta M \varphi(t) = M_R[1 + \Delta\varphi(t)]$$

$$\varphi(0) = 1 \quad ; \quad \varphi(\infty) = 0$$

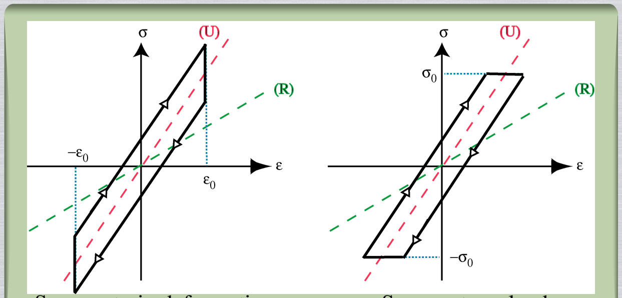
- So that

$$J_R = J_U(1+\Delta) \quad ; \quad M_U = M_R(1+\Delta)$$

Linearity?

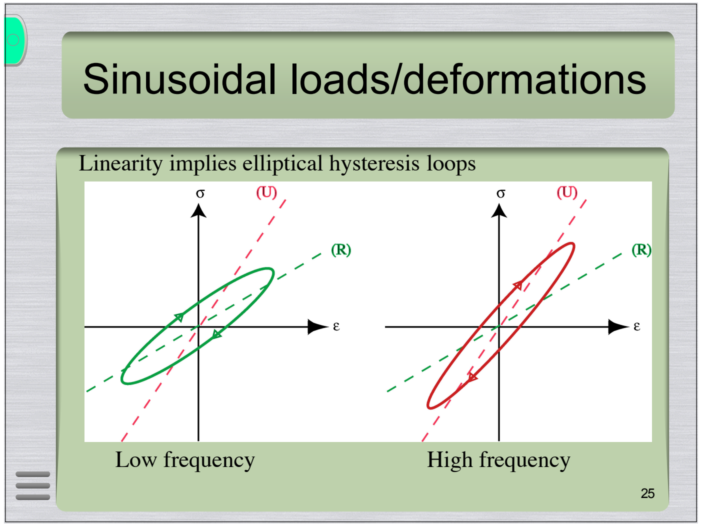
- (large) 1mm amplitude wave for 10km wavelength gives strain O(10^{-7})
- Laboratory data shows rocks to be linear for strains < O(10^{-6})
- Losses measured by area of hysteresis loops.

Periodic Square-Wave hysteresis



Square strain deformation

Square stress load



Sinusoidal loads/deformations

Linearity implies elliptical hysteresis loops

Low frequency

High frequency

Hysteresis implications

- The modulus (or compliance) depends on frequency.
  - Stiff material at high frequencies
  - Compliant material at low frequencies
- Wave velocity depends therefore on frequency: **Physical Dispersion**

Part 2: Thermodynamics

Balance Equations

- Away from flow singularities (eg discontinuities) a quantity of the flow *F* is conserved if:

$$\frac{\partial F}{\partial t} + \nabla_x \bullet (F\mathbf{V} - \mathbf{J}) = K$$

Where ***V*** is the particle velocity at spatial position *x*, ***FV*** is a “convection” (advection) flux,***J*** is a “conduction” flux, and *K* is an intrinsic source strength

Conservation Laws $\frac{\partial F}{\partial t} + \nabla_x \bullet (F\mathbf{V} - \mathbf{J}) = K$

Law	F	$\mathbf{J}$	K
Mass	ρ	0	0
Momentum	$\rho\mathbf{V}$	$\mathbf{T}$	$\rho\mathbf{b}$
Energy	ρE	$\mathbf{T}\bullet\mathbf{V} - \mathbf{q}$	$\rho(\mathbf{b}\bullet\mathbf{V} + h)$

$\rho =$ *density*

$\mathbf{V} =$ *particle velocity*

$\mathbf{T} =$ *Cauchy stress tensor*

$E = u + \frac{1}{2}\mathbf{V}\bullet\mathbf{V}$

$\mathbf{b} =$ *body force density*

$\mathbf{q} =$ *heat flux vector*

$h =$ *heat source density*

$u =$ *internal energy density*

Internal Energy balance

- By combination of conservation laws:

$$\frac{\partial \rho u}{\partial t} + (\rho u V_i + q_i)_{,i} = T_{ij} D_{ij} + \rho h$$

- Where $D_{ij} = (V_{i.j} + V_{j.i})/2$
  is the **rate of deformation** tensor
- Equations may be generalized to include other, nonmechanical fluxes (e.g. electrical, etc.) **q** must then be generalized as well.

Linearized Near-Equilibrium Thermodynamics (LNET)

- Postulates:

1. Local thermodynamic equilibrium
2. Linearity of phenomenological equations
3. Onsager-Casimir symmetry relations (invariance of equations of motions under time reversal)

L.N.E.T. Postulate 1

- Even though a system may not be in a state of global equilibrium, infinitesimal elements may be in local equilibrium, *and the functional dependence of state functions on state parameters is the same as in the case of equilibrium in classical thermodynamics.*
- Leads to **Gibbs relations**

Caloric Equation of State

- Energy density determined by thermodynamic state, ie entropy density and state variables

$$u = u(s, \nu_1, \ldots, \nu_m, \xi_1, \ldots, \xi_n)$$

- Macroscopically observable variables and conjugate thermodynamic tensions:

$$\nu_i,\ i = 1, \ldots, m \quad conjugate\ to\ \tau_i,\ \ i = 1, \ldots, m$$

- Internal (unobservable) variables, and associated affinities

$$\xi_j,\ \ j = 1, \ldots, n \quad associated\ with\ A_j,\ \ j = 1, \ldots, n$$

Gibbs relations

- Gibbs relation reads

$$du = Tds + \sum_{i=1}^{m} \tau_i d\nu_i - \sum_{j=1}^{n} A_j d\xi_j$$

- T is temperature, s is entropy. For locally adiabatic processes, then

$$du = \boldsymbol{\tau} \bullet d\boldsymbol{\nu} - \mathbf{A} \bullet d\boldsymbol{\xi}$$

Thermal Equations of State

- Expand tensions and affinities near equilibrium

$$\begin{bmatrix} \boldsymbol{\tau} \\ -\mathbf{A} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\tau}^0 \\ -\mathbf{A}^0 \end{bmatrix} + \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{\nu} - \boldsymbol{\nu}^0 \\ \boldsymbol{\xi} - \boldsymbol{\xi}^0 \end{bmatrix}$$

- Equilibrium values (0 superscript) will be set to zero (WLOG)

Maxwell's Relations

- u is a perfect differential. The order of differentiation is immaterial, so the matrix **P** is symmetric:

$$\mathbf{P}_{11} = \mathbf{P}_{11}^T \quad ; \quad \mathbf{P}_{22} = \mathbf{P}_{22}^T \quad ; \quad \mathbf{P}_{12} = \mathbf{P}_{12}^T$$

- **Stable equilibrium**: *u* must be a minimum for given $\boldsymbol{\tau}$, **A.** So **P** must be positive-definite.

Nota bene..

- τ are **thermodynamic** tensions for which *u* is a potential. If ν are strains, then τ are stresses only if the external stress work is recoverable (eg adiabatic or isothermal processes with resersible heat transfer)
- In general, the dynamic stress tensor is not identifiable with the thermodynamic tensions. The difference is called the **viscous stress tensor.**

Entropy balance (1)

- Introduce dynamic tensions $\boldsymbol{\tau}^D$ as the observable quantities,then

$$\frac{\partial \rho u}{\partial t}+(\rho u V_i+q_i)_{,i}=T_{ij}D_{ij}+\rho h$$

is of the general form:

$$\rho\frac{du}{dt}=-\nabla\bullet\mathbf{q}+K_u \quad where \quad K_u=\sum_{i=1}^{m}\tau_i^D\dot{v}_i$$

- Hence from the Gibbs relation, the entropy balance reads:

Entropy balance (2)

$$\rho\frac{ds}{dt}=-\nabla\bullet\frac{\mathbf{q}}{T}+K_s$$

- Where the entropy source strength is

$$K_s=\frac{1}{T}\left(\boldsymbol{\tau}^D-\boldsymbol{\tau}\right)\bullet\dot{\mathbf{v}}+\frac{1}{T}\mathbf{A}\bullet\dot{\boldsymbol{\xi}}-\frac{\nabla T\bullet\mathbf{q}}{T^2}$$

- The **Dissipation function** is

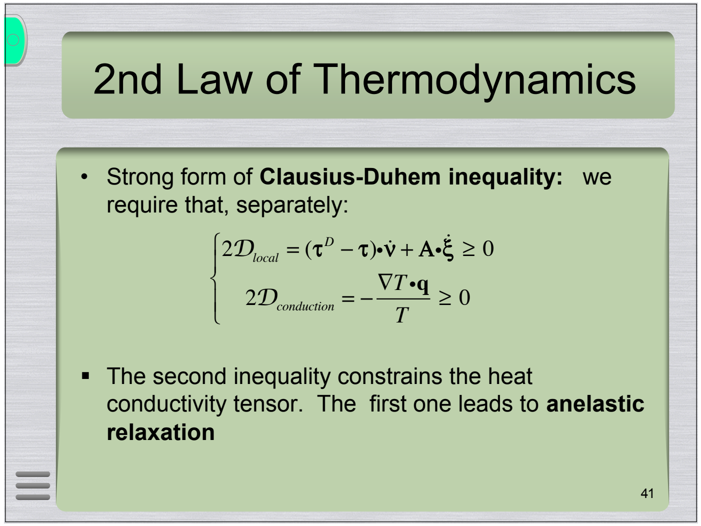
$$\mathcal{D}=\frac{1}{2}TK_s \quad of\ the\ general\ form\ \mathcal{D}=\frac{1}{2}\sum_i \mathrm{X}_i H_i$$

Entropy balance (3)

$$\mathcal{D}=\frac{1}{2}\sum_i \mathrm{X}_i H_i$$

By comparison with the earlier expressions, X_i are called **generalized thermodynamic forces** and H_i **generalized fluxes**

The **2nd Law** requires that $\mathcal{D}$ be a nonnegative function that vanishes only for reversible processes. This is expressed through the **Clausius-Duhem inequality**

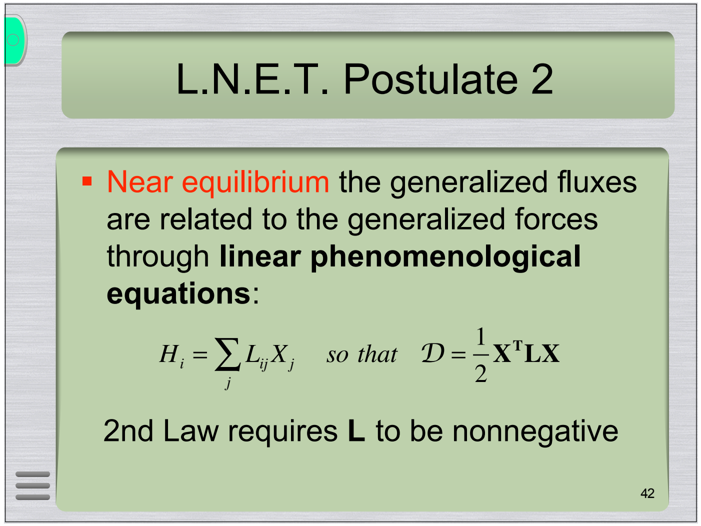


2nd Law of Thermodynamics

- Strong form of **Clausius-Duhem inequality:** we require that, separately:

$$\begin{cases} 2\mathcal{D}_{local} = (\boldsymbol{\tau}^{D} - \boldsymbol{\tau})\bullet\dot{\mathbf{v}} + \mathbf{A}\bullet\dot{\boldsymbol{\xi}} \geq 0 \\ 2\mathcal{D}_{conduction} = -\dfrac{\nabla T\bullet\mathbf{q}}{T} \geq 0 \end{cases}$$

- The second inequality constrains the heat conductivity tensor. The first one leads to **anelastic relaxation**

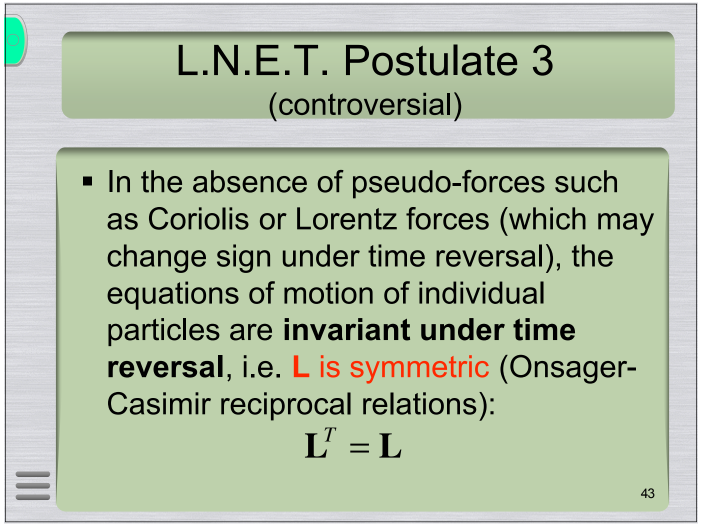


L.N.E.T. Postulate 2

- Near equilibrium the generalized fluxes are related to the generalized forces through **linear phenomenological equations**:

$$H_i = \sum_j L_{ij} X_j \quad \textit{so that} \quad \mathcal{D} = \frac{1}{2}\mathbf{X}^{\mathrm{T}}\mathbf{L}\mathbf{X}$$

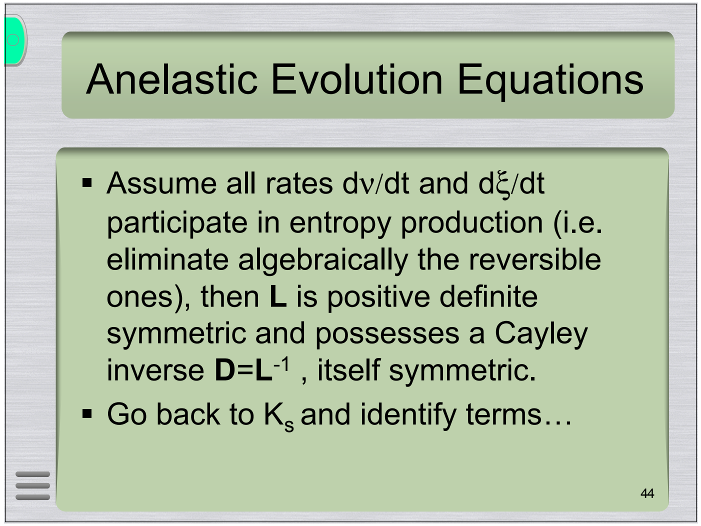
2nd Law requires **L** to be nonnegative



L.N.E.T. Postulate 3
(controversial)

- In the absence of pseudo-forces such as Coriolis or Lorentz forces (which may change sign under time reversal), the equations of motion of individual particles are **invariant under time reversal**, i.e. **L** is symmetric (Onsager-Casimir reciprocal relations):

$$\mathbf{L}^{T} = \mathbf{L}$$



Anelastic Evolution Equations

- Assume all rates dν/dt and dξ/dt participate in entropy production (i.e. eliminate algebraically the reversible ones), then **L** is positive definite symmetric and possesses a Cayley inverse **D**=**L**$^{-1}$, itself symmetric.
- Go back to K_s and identify terms…

Anelastic Evolution Equations

$$K_s = \frac{1}{T}\left(\boldsymbol{\tau}^D - \boldsymbol{\tau}\right)\bullet\dot{\boldsymbol{\nu}} + \frac{1}{T}\mathbf{A}\bullet\dot{\boldsymbol{\xi}} - \frac{\nabla T \bullet \mathbf{q}}{T^2}$$

$$\begin{bmatrix} \boldsymbol{\tau} \\ -\mathbf{A} \end{bmatrix} = -\begin{bmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{D}_{12}^T & \mathbf{D}_{22} \end{bmatrix}\begin{bmatrix} \dot{\boldsymbol{\nu}} \\ \dot{\boldsymbol{\xi}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\tau}^D \\ 0 \end{bmatrix}$$

Combine with Thermal Equations of State, viz

$$\begin{bmatrix} \boldsymbol{\tau} \\ -\mathbf{A} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\tau}^0 \\ -\mathbf{A}^0 \end{bmatrix} + \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix}\begin{bmatrix} \boldsymbol{\nu} - \boldsymbol{\nu}^0 \\ \boldsymbol{\xi} - \boldsymbol{\xi}^0 \end{bmatrix}$$

Anelastic Evolution Equations

$$\begin{bmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{D}_{12}^T & \mathbf{D}_{22} \end{bmatrix}\begin{bmatrix} \dot{\boldsymbol{\nu}} \\ \dot{\boldsymbol{\xi}} \end{bmatrix} + \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{12}^T & \mathbf{P}_{22} \end{bmatrix}\begin{bmatrix} \boldsymbol{\nu} \\ \boldsymbol{\xi} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\tau}^D \\ 0 \end{bmatrix}$$

adopt a common notation χ for **all** state variables (macro and internal), and define the quadratic form

$$\mathcal{P} = \frac{1}{2}\boldsymbol{\chi}^{\mathrm{T}}\mathbf{P}\boldsymbol{\chi}$$

we find the familiar Lagrangean equations of evolution

$$\boxed{\frac{\partial \mathcal{D}}{\partial \dot{\chi}_i} + \frac{\partial \mathcal{P}}{\partial \chi_i} = \mathcal{Q}_i}$$

Where χ_i are generalized displacements and $\mathcal{Q}$i are generalized forces, and $\mathcal{P}$ a potential function

Solving the equations (1)

- Assume zero initial conditions,and take Laplace transforms $F(t) \to \bar{F}(s)$

$$\begin{bmatrix} s\mathbf{D}_{11} + \mathbf{P}_{11} & s\mathbf{D}_{12} + \mathbf{P}_{12} \\ s\mathbf{D}_{12}^T + \mathbf{P}_{12}^T & s\mathbf{D}_{22} + \mathbf{P}_{22} \end{bmatrix}\begin{bmatrix} \bar{\boldsymbol{\nu}} \\ \bar{\boldsymbol{\xi}} \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{\tau}}^D \\ 0 \end{bmatrix}$$

2nd Law requires $\mathbf{D}_{22}$ to be positive definite, so $\mathbf{D}_{22}$ and $\mathbf{P}_{22}$ can be simultaneously diagonalized.

Solving the equations (2)

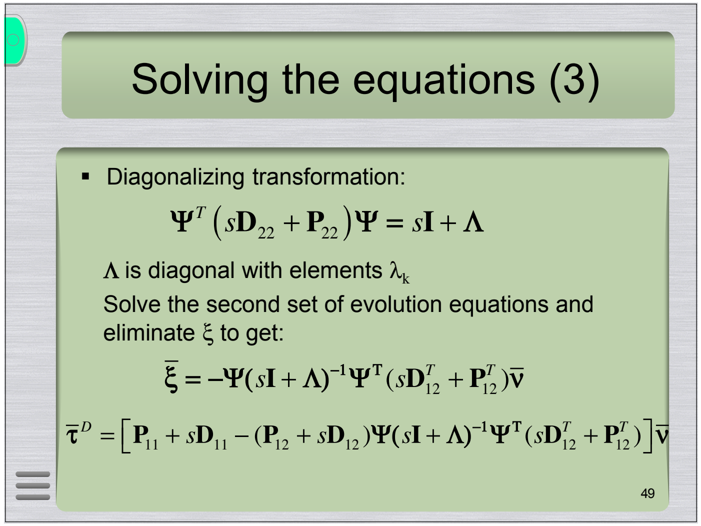
- Solve the generalized eigenvalue problem

$$\mathbf{P}_{22}\boldsymbol{\xi} - \lambda\mathbf{D}_{22}\boldsymbol{\xi} = 0$$

n eigenvalues λ_k , k = 1,…,n, are solutions of the secular equation:

$$\det\left|\mathbf{P}_{22} - \lambda\mathbf{D}_{22}\right| = 0$$

The corresponding n eigenvectors are called **relaxation modes**; they are **D**$_{22}$-orthonormal linear combinations of the (coupled) state variables, and are the columns of a matrix $\boldsymbol{\Psi}$ defining the diagonalizing transformation.



Solving the equations (3)

- Diagonalizing transformation:

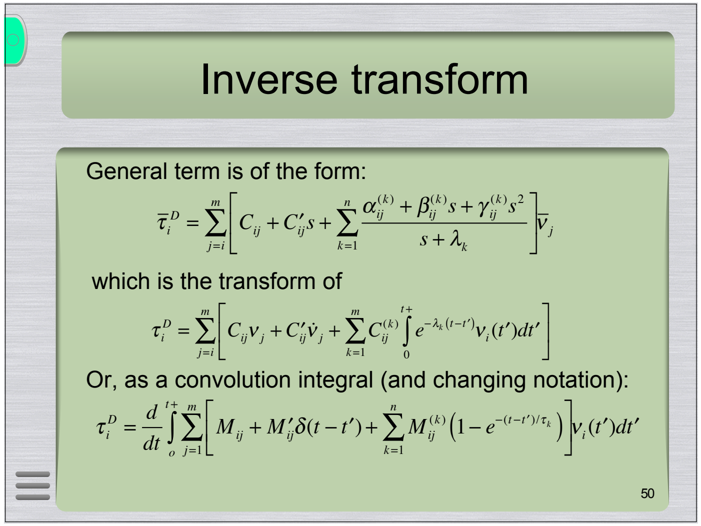
$$\boldsymbol{\Psi}^T\left(s\mathbf{D}_{22}+\mathbf{P}_{22}\right)\boldsymbol{\Psi}=s\mathbf{I}+\boldsymbol{\Lambda}$$

$\boldsymbol{\Lambda}$ is diagonal with elements λ_k

Solve the second set of evolution equations and eliminate ξ to get:

$$\bar{\boldsymbol{\xi}}=-\boldsymbol{\Psi}(s\mathbf{I}+\boldsymbol{\Lambda})^{-1}\boldsymbol{\Psi}^{\mathbf{T}}(s\mathbf{D}_{12}^T+\mathbf{P}_{12}^T)\bar{\mathbf{v}}$$

$$\bar{\boldsymbol{\tau}}^D=\left[\mathbf{P}_{11}+s\mathbf{D}_{11}-(\mathbf{P}_{12}+s\mathbf{D}_{12})\boldsymbol{\Psi}(s\mathbf{I}+\boldsymbol{\Lambda})^{-1}\boldsymbol{\Psi}^{\mathbf{T}}(s\mathbf{D}_{12}^T+\mathbf{P}_{12}^T)\right]\bar{\mathbf{v}}$$



Inverse transform

General term is of the form:

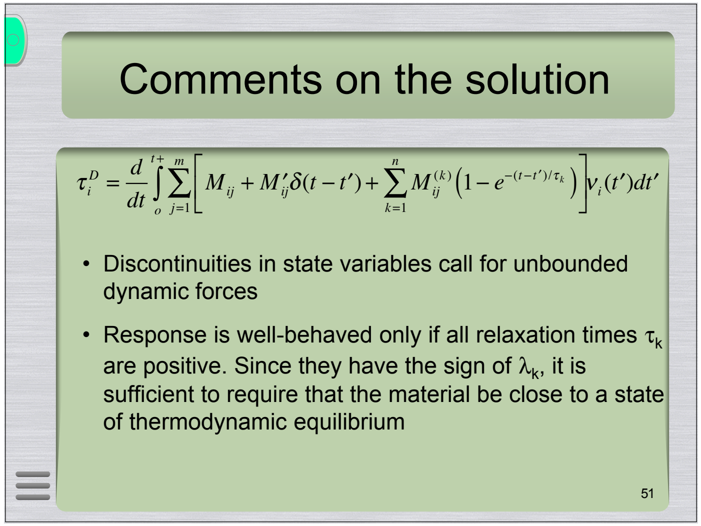
$$\bar{\tau}_i^D=\sum_{j=i}^{m}\left[C_{ij}+C'_{ij}s+\sum_{k=1}^{n}\frac{\alpha_{ij}^{(k)}+\beta_{ij}^{(k)}s+\gamma_{ij}^{(k)}s^2}{s+\lambda_k}\right]\bar{v}_j$$

which is the transform of

$$\tau_i^D=\sum_{j=i}^{m}\left[C_{ij}v_j+C'_{ij}\dot{v}_j+\sum_{k=1}^{m}C_{ij}^{(k)}\int_0^{t+}e^{-\lambda_k(t-t')}v_i(t')dt'\right]$$

Or, as a convolution integral (and changing notation):

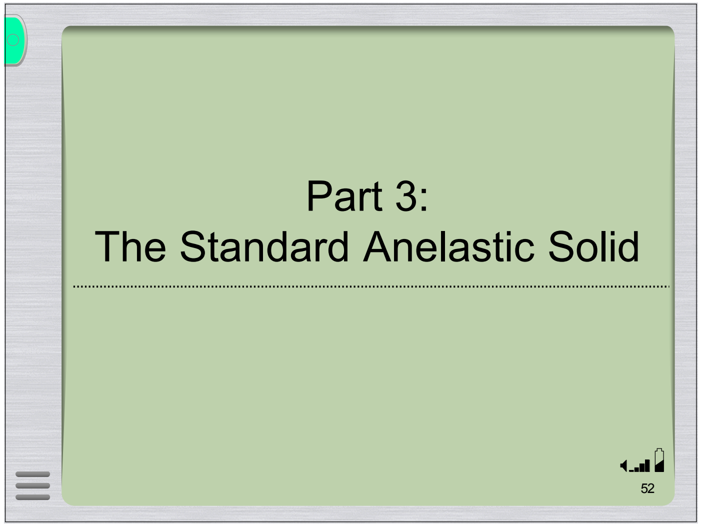
$$\tau_i^D=\frac{d}{dt}\int_o^{t+}\sum_{j=1}^{m}\left[M_{ij}+M'_{ij}\delta(t-t')+\sum_{k=1}^{n}M_{ij}^{(k)}\left(1-e^{-(t-t')/\tau_k}\right)\right]v_i(t')dt'$$



Comments on the solution

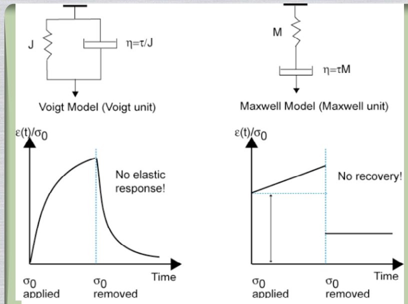
$$\tau_i^D=\frac{d}{dt}\int_o^{t+}\sum_{j=1}^{m}\left[M_{ij}+M'_{ij}\delta(t-t')+\sum_{k=1}^{n}M_{ij}^{(k)}\left(1-e^{-(t-t')/\tau_k}\right)\right]v_i(t')dt'$$

- Discontinuities in state variables call for unbounded dynamic forces
- Response is well-behaved only if all relaxation times τ_k are positive. Since they have the sign of λ_k, it is sufficient to require that the material be close to a state of thermodynamic equilibrium



Part 3:
The Standard Anelastic Solid

Voigt and Maxwell



$$J\sigma = \varepsilon + \tau\dot{\varepsilon} \qquad \tau\dot{\sigma} + \sigma = \tau M\dot{\varepsilon}$$

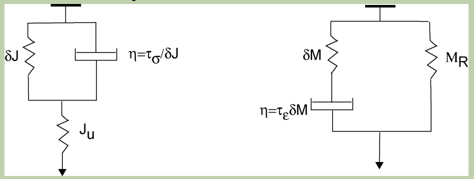
- Two-parameter mechanical models
- Neither fully satisfactory

Three-parameter Models

General linear differential stress-strain equation with constant coefficients:

$$a_0\sigma + a_1\dot{\sigma} + a_2\ddot{\sigma} + ... = b_0\varepsilon + b_1\dot{\varepsilon} + b_2\ddot{\varepsilon}....$$

Case where only first time derivatives enter:



$$J_R\sigma + J_U\tau_\sigma\dot{\sigma} = \varepsilon + \tau_\sigma\dot{\varepsilon} \qquad \sigma + \tau_\varepsilon\dot{\sigma} = M_R\varepsilon + M_U\tau_\varepsilon\dot{\varepsilon}$$

Standard Anelastic Solid

$$J_R\sigma + J_U\tau_\sigma\dot{\sigma} = \varepsilon + \tau_\sigma\dot{\varepsilon} \qquad \sigma + \tau_\varepsilon\dot{\sigma} = M_R\varepsilon + M_U\tau_\varepsilon\dot{\varepsilon}$$

Integrate…for a single relaxation time, we get:

$$\sigma(t) = M_U\varepsilon(t) - \delta M\int_0^t\left(1 - e^{-(t-t')/\tau_\varepsilon}\right)\varepsilon(t')dt'$$

Instantaneous elastic response + **exponentially fading memory**

Generalization to Spectrum of Relaxation Times

Recall: $\begin{cases} J(t) = J_U\left[1+\Delta\psi(t)\right] \\ M(t) = M_R\left[1+\Delta\varphi(t)\right] \end{cases}$

Compliance response (to unit step stress)
Relaxation response (to unit step strain)
Then define the **normalized retardation spectrum** Ψ and **normalized relaxation spectrum** Φ by:

$$\begin{cases} \psi(t) = 1 - \int_{-\infty}^{\infty}\Psi(\ln\tau)e^{-t/\tau}d(\ln\tau) \\ \varphi(t) = \int_{-\infty}^{\infty}\Phi(\ln\tau)e^{-t/\tau}d(\ln\tau) \end{cases}$$

Bolztmann After-Effect equations

- Linearity means that the responses may be cast as convolutions cast as Stieltjes integrals (superposition principle) :

$$\begin{cases} \varepsilon(t) = J(t) * d\sigma(t) = \int_{-\infty}^{t+} J(t-t')d\sigma(t') \\ \sigma(t) = M(t) * d\varepsilon(t) = \int_{-\infty}^{t+} M(t-t')d\varepsilon(t') \end{cases}$$

Bolztmann After-Effect equations

- For a medium initially at rest and in equilibrium:

$$\begin{cases} \varepsilon(t) = \dfrac{d}{dt}\int_{0}^{t+} J(t-t')\sigma(t')dt' \\ \sigma(t) = \dfrac{d}{dt}\int_{0}^{t+} M(t-t')\varepsilon(t')dt' \end{cases}$$

Current state depends on all previous history with an exponentially fading memory

Use of Transforms

- Apply “Carlson”, or s-multiplied Laplace transform

$$\varepsilon^*(s) = s\int_0^\infty e^{-st}\varepsilon(t)dt \qquad \sigma^*(s) = s\int_0^\infty e^{-st}\sigma(t)dt \quad .... \; etc$$

Then, the Boltzmann after-effect equations read:

$$\begin{cases} \varepsilon^*(s) = J^*(s)\sigma^*(s) \\ \sigma^*(s) = M^*(s)\varepsilon^*(s) \end{cases}$$

Looks functionally just like linear elasticity! This is known as the **Correspondence Principle**

Use of Transforms

J*(s) and M*(s) are known as the **operational** compliance and **operational** modulus, respectively.

If we effect the change of variable from s to iω, we get the **complex** compliance J*(iω) and **complex** modulus M*(iω), which are useful for analysis in the frequency domain (Fourier instead of Laplace transforms).

We have the limits:

$$\lim_{s\to 0}\begin{pmatrix} M^*(s) \\ J^*(s) \end{pmatrix} = \lim_{\omega\to 0}\begin{pmatrix} M^*(i\omega) \\ J^*(i\omega) \end{pmatrix} = \begin{pmatrix} M_R \\ J_R \end{pmatrix} \quad ; \quad \lim_{s\to \infty}\begin{pmatrix} M^*(s) \\ J^*(s) \end{pmatrix} = \lim_{\omega\to \infty}\begin{pmatrix} M^*(i\omega) \\ J^*(i\omega) \end{pmatrix} = \begin{pmatrix} M_U \\ J_U \end{pmatrix}$$

General 3-D Case

In the general case (follow the treatment of elasticity):

$$\begin{cases} \varepsilon_{ij} = J_{ij}^{kl} * d\sigma_{kl} = \sigma_{kl} * dJ_{ij}^{kl} \\ \sigma_{ij} = M_{ij}^{kl} * d\varepsilon_{kl} = \varepsilon_{kl} * dM_{ij}^{kl} \end{cases}$$

The same symmetries apply to the operational compliance and modulus tensors as for the elastic tensor. Similarly, for an isotropic material we only have two creep functions, or two relaxation functions: bulk and deviatoric.
Etc…

Part 5:
Waves and Vibrations

Q

- Energy (potential or kinetic) is transformed into heat. Following engineering convention we introduce the "**quality factor**" Q by:

$$Q^{-1} = \Delta E / 2\pi E_{peak}$$

where E_{peak} is the peak stored energy in a given volume and ΔE the energy dissipated per cycle in the same volume. This is OK for small losses(e.g. electrical circuits), but runs into problems for Q<6 … All the energy is lost in the first cycle!

Waves in a lossy medium

Instead, we introduce <E> the average stored energy in a cycle, and define:

$$\boxed{Q^{-1} = \Delta E / 4\pi \langle E \rangle}$$

In the limit of low loss the definitions are equivalent. Consider monochromatic stress-strain histories,i.e.

$$\sigma(t) = \mathrm{Re}\left[\sigma_0 e^{i\omega t}\right] \quad ; \quad \varepsilon(t) = \mathrm{Re}\left[\varepsilon_0 e^{i\omega t}\right]$$

Then the (complex) amplitudes are related by:

$$\varepsilon_0 = J^*(i\omega)\sigma_0 \quad ; \quad \sigma_0 = M^*(i\omega)\varepsilon_0$$

Expressions for Q

Energy loss per cycle, and average energy in cycle are:

$$\begin{cases} \Delta E(\omega) = \oint \sigma d\varepsilon = -\pi |\sigma_0|^2 \operatorname{Im}[J^*(\omega)] = \pi |\varepsilon_0|^2 \operatorname{Im}[M^*(\omega)] \\ \langle E \rangle = \frac{1}{4} |\sigma_0|^2 \operatorname{Re}[J^*(\omega)] = \frac{1}{4} |\varepsilon_0|^2 \operatorname{Re}[M^*(\omega)] \end{cases}$$

So that we get the remarkably simple result:

$$\boxed{\begin{cases} J^*(i\omega) = J_1(\omega) - iJ_2(\omega) \\ M^*(i\omega) = M_1(\omega) + iM_2(\omega) \end{cases} \quad then \quad Q^{-1} = \frac{M_2(\omega)}{M_1(\omega)} = \frac{J_2(\omega)}{J_1(\omega)}}$$

Notes

The real parts M_1 and J_1 are called the **storage** modulus and compliance, and are monotonic functions of frequency, from M_R and J_R at long periods to M_U and J_U high frequencies.

The imaginary parts M_2 and J_2 are called the **loss** modulus and compliance, show a maximum at intermediate frequencies, and vanish at low and high frequencies.

The phase difference between stress and strain is given by:

$$\tan\varphi = Q^{-1}(\omega)$$

φ is known as the **loss angle.** It is frequency-dependent

Advanced notes…

- Because of frequency dependence of the complex modulus and compliance, there is physical dispersion.
- In the time domain J(t) and M(t) are required to be "**causal"**, so effects do not precede causes. They must vanish for t<0. This requires that the **Kramers-Krönig** relations be satisfied, namely that the real and imaginary parts of J* and M* be Hilbert transform pairs.

$$M_1(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{M_2(\varsigma)}{\varsigma - \omega} d\varsigma \quad ; \quad M_2(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{M_1(\varsigma)}{\varsigma - \omega} d\varsigma \quad ;$$

Key Point: wave propagation speed and attenuation are not independent functions of frequency

Simple Dynamical Problems

Consider a one-dimensional wave propagation problem

$$\rho \frac{\partial^2 u}{\partial t^2} = \rho \ddot{u} = \frac{\partial \sigma}{\partial x}, \; where \;\; \sigma(t) = \frac{d}{dt} \int_0^{t+} M(t - t')\varepsilon(t')dt', \;\; and \;\; \varepsilon = \frac{\partial u}{\partial x} = u''$$

Take transforms,and specialized to monochromatic case:

$$\begin{cases} \rho s^2 u^* - M^*(s) u''^* = 0 \\ (M_1 + iM_2)\overline{u}'' + \rho\omega^2 \overline{u} = 0 \end{cases}$$

(Asterisks denote Laplace transforms, bars Fourier transforms). Introduce the **complex wave number**:

$$k^2(\omega) = \rho\omega^2 / (M_1 + iM_2) = \rho\omega^2 / M_1(1 + iQ^{-1}) \,, \quad \operatorname{Re} k \geq 0$$

Simple Dynamical Problems

The wave equation becomes:

$$\bar{u}'' + k^2(\omega)\bar{u} = 0, \quad where \;\; k(\omega) = \frac{\omega}{c(\omega)} - i\alpha(\omega)$$

C(ω) is the **phase velocity**, α(ω) is the **attenuation coefficient**. For low loss, ($Q^{-1} << 1$)

$$\begin{cases} c(\omega) \simeq \sqrt{M_1(\omega)/\rho} \\ \alpha(\omega) \simeq \omega Q^{-1}(\omega)/2c(\omega) \end{cases}$$

Forced Resonance

Seek steady state solution with boundary conditions:

$$\begin{cases} u \equiv 0 & at \quad x = 0 \\ u = u_0 \sin\omega t & at \quad x = L \end{cases}$$

The solution $u = u_0 \sin\omega t \dfrac{\sin[k(\omega)x]}{\sin[k(\omega)L]}$

has maxima in the low-loss limit near the unattenuated resonance frequencies defined by

$$\omega_n / c(\omega_n) = n\pi / L$$

Forced Resonance

At resonance, the denominator is:

$$\left|\sin k(\omega_n)L\right|^2 \simeq \alpha^2(\omega_n)L^2 \simeq n^2\pi^2/4Q^2$$

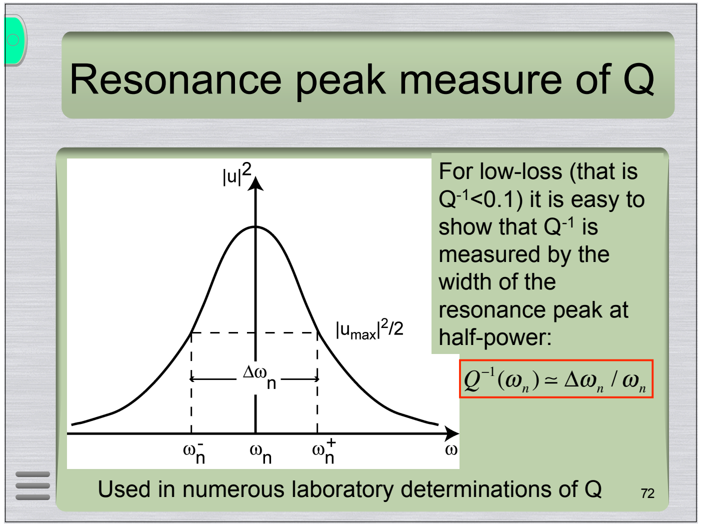
So the shape of the resonance peak is given by

$$|u|^2 \propto \left[\left(1 - \alpha^2 L^2\right)\sin^2\frac{(\omega - \omega_n)L}{c(\omega_n)} + \alpha^2 L^2\right]^{-1}$$

which can be approximated for **low-loss** by the "Lorentzian peak" form

$$|u|^2 \propto \left[(\omega - \omega_n)^2 + \frac{Q^{-2}}{4 - n^2\pi^2 Q^{-2}}\omega_n^2\right]^{-1}$$

Resonance peak measure of Q



For low-loss (that is $Q^{-1}<0.1$) it is easy to show that Q^{-1} is measured by the width of the resonance peak at half-power:

$$\boxed{Q^{-1}(\omega_n) \simeq \Delta\omega_n / \omega_n}$$

Used in numerous laboratory determinations of Q

Free vibrations

Seek solution satisfying boundary conditions:

$$u \equiv 0 \quad at \quad x = 0 \quad and \quad x = L$$

And initially vibrating at one of the eigenfrequencies ω_n .
Seek solutions of form:

$$u = u_n(t)\sin\frac{n\pi x}{L} = u_n(t)\sin\frac{\omega_n x}{c(\omega_n)}$$

Then the wave equation: $$\bar{u}'' + k^2(\omega)\bar{u} = 0$$

only has a nontrivial solution if $$\boxed{k^2(\omega) - \frac{\omega_n^2}{c^2(\omega_n)} = 0}$$

Low-loss free vibrations

For low loss, and ω close to ω_n, expressions for k yield:

$$\omega = \omega_n(1 + iQ^{-1}/2)$$

So that

$$u(t) \propto \exp(i\omega_n t)\exp\left[-\frac{\omega_n t}{2Q(\omega_n)}\right]\sin\frac{\omega_n x}{c(\omega_n)}$$

The amplitude decays exponentially, with a logarithmic decrement given by

$$\delta = \ln\left[\frac{u_n(t)}{u_n(t + 2\pi/\omega_n)}\right] = \pi Q^{-1}(\omega_n)$$

Q estimates from free vibrations

Take transform:

$$\bar{u}(\omega) = \int_0^\infty e^{i\omega t}u_n(t)dt = \frac{1}{\alpha_n - i(\omega - \omega_n)} \quad ; \quad \alpha_n = \omega_n / 2Q(\omega_n)$$

Power spectrum:

$$|\bar{u}(\omega)|^2 \propto \left[\alpha_n^2 + (\omega_n - \omega)^2\right]^{-1}$$

This is the same as in the case of forced vibrations!!
Measure Q from the width of resonance peaks:

$$\boxed{Q^{-1}(\omega_n) \simeq \Delta\omega_n / \omega_n}$$

Traveling waves

The monochromatic traveling wave solution is of the form:

$$u = u_0 \exp\left[i\omega t - ik(\omega)x\right]$$

Again, note that the wave number is now complex so

$$u = u_0 \exp\left[i\omega(t - x/c(\omega))\right]\exp\left[-\alpha(\omega)x\right]$$

Thus α(ω) measures the amplitude decay as a function of distance traveled. It has dimension $[L]^{-1}$and is expressed in decibel / unit length.

Traveling waves

In terms of the wavelength λ, we get, for low loss:

$$\boxed{u = u_0\left[i\omega[t - x/c(\omega)]\right]\exp[-\pi x/Q\lambda]}$$

The amplitude is thus controlled by the number of wavelengths traveled by the wave. For a fixed path length x, a broad-band pulse, and a frequency-independent Q model, the high frequency components (short wavelengths) are more attenuated than the long-period components (long wavelengths).

Summary: low-loss results

For $Q^{-1} \ll 1$ we have:

$$\boxed{Q^{-1} = \tan\varphi \simeq \varphi \simeq \delta/\pi \simeq \alpha\lambda/\pi}$$

Near a resonance peak at frequency ω_n with peak width $\Delta\omega_n$

$$\boxed{Q^{-1} \simeq \Delta\omega_n/\omega_n}$$

Measurements of ω_n for forced or free vibrations, or measurements of the phase velocity c(ω) in traveling wave experiments yield an estimate of the real part of the modulus M(ω) *at the frequency ω*.

Part 6:
Absorption bands

Seismic problems

- Seismic pulses tend to be broad-band
- Q is observed to depend little on frequency over a broad range of frequencies
- We need to specify a broad relaxation (or retardation) spectrum, or the creep (or relaxation) function.
- What is usually done is to specify Q(ω)

A broad-band Q model

Start with “*Jeffrey’s modified Lomnitz law of creep*”

$$\Delta\psi(t) = \frac{q}{\alpha}\left[\left(1+\frac{t}{\tau_m}\right)^{\alpha} - 1\right]$$

Then the wavenumber is expressed as

$$k^2(\omega) = \frac{\omega^2}{V_U^2}\left[1 - q\frac{e^{i\omega\tau_m}}{(i\omega\tau_m)^{\alpha}}\Gamma(i\omega\tau_m)\right]$$

V_u is the unrelaxed wave speed, and Γ is an incomplete Gamma function. We have basically two frequency regions.

A broad-band Q model

a) At long periods $\omega\tau_m << 1$

$$Q(\omega) \simeq \cot\frac{\alpha\pi}{2} + \frac{(1+q/\alpha)(\omega\tau_m)^{\alpha}}{q\Gamma(\alpha)\sin\frac{\alpha\pi}{2}} \quad ; \quad for \;\; small \;\; \alpha,\ Q(\omega) \simeq q\pi/2$$

b) At high frequencies $\omega\tau_m >> 1$

$$Q(\omega) \simeq \omega\tau_m / q$$

Typical range for α is [0.1, 0.3], yielding weak frequency dependence at low frequencies, and strong dependence at high frequencies.

Pulse scaling

We seek to solve a pulse propagation problem with

$$\begin{cases} u(t,0) = \delta(t) \\ \lim\limits_{x\to\infty} u(t,x) = 0 \end{cases}$$

We can write the solution and an inverse Fourier transform

$$u(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} e^{-ik(\omega)x}\, d\omega$$

This can be evaluated numerically, with care!

Pulse Scaling

Change time origin to the unrelaxed travel time and consider the low loss approximation:

$$t_U = x/V_U \quad ; \quad \tau = t - t_U$$

$$u(t,x) \simeq \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau} \exp\left[-i\omega t_U \frac{q e^{i\omega\tau_m}}{2(i\omega\tau_m)^{\alpha}}\Gamma(\alpha, i\omega\tau_m)\right] d\omega$$

When q is small and τ_m not too large, then

$$u(t,x) \simeq \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau} \exp\left[\frac{-i\omega t_U q\Gamma(\alpha)}{2(i\omega\tau_m)^{\alpha}}\right] d\omega$$

Scaling parameters

Define $\tau'_m = a\tau_m \;;\; q' = bq \;;\; t'_U = ct_U \;;\; \tau' = d\tau$

We obtain the scaling law for time variables

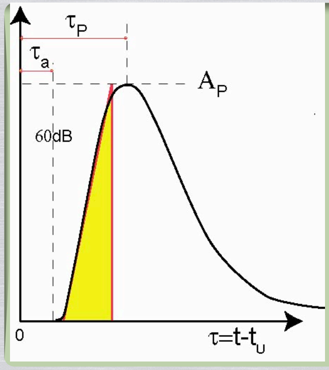
$$\boxed{[T] \propto f(\alpha)P \quad ; \quad P = (qt_U\tau_m^{-\alpha})^{1/(1-\alpha)}}$$

…and for amplitudes

$$\boxed{[A] \propto f^{-1}(\alpha)P^{-1}}$$

The scaling function f(α) is a decreasing function of α

Q operator



For low loss:
-the arrival time is slightly later than the unrelaxed arrival time ("pedestal")
- i.e. the "signal" velocity is less than the unrelaxed velocity
- The colored area is preserved during propagation. Amplitudes decay, times lengthen.

Q Operator scaling

The scaling laws do not apply if Q is very small, strongly frequency-dependent, or if the characteristic time in the creep law is comparable to the travel time.

In the limit of α very small, Q(ω) becomes nearly constant for low enough frequency.
For $\omega < \tau_m^{-1}$ we have the limiting scaling law:

$$\boxed{[T] \propto f(0)\frac{2}{\pi}\frac{t_U}{Q} = f(0)\frac{2}{\pi}t^* \simeq \frac{1}{2}t^*}$$

The parameter t* is commonly used in seismology. It assumes implicitly that Q is frequency independent.

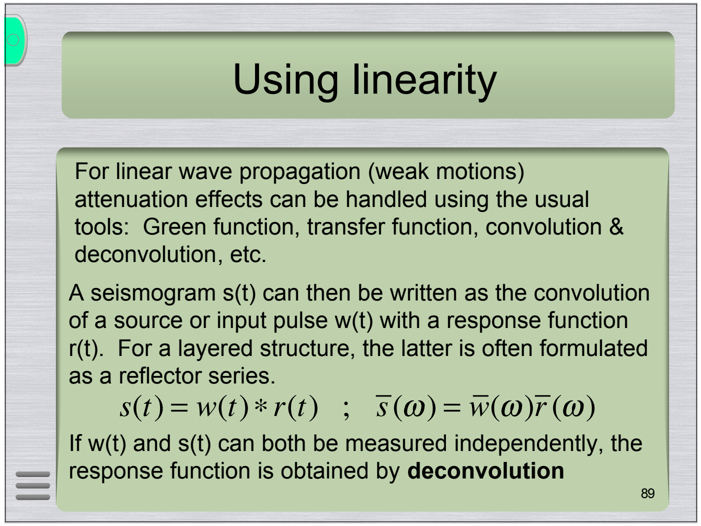
Propagation

We have seen that body wave attenuation leads to a scaling of the pulses controlled by the parameter:

$$t^* = t_U / Q = \frac{Travel\ \ time}{Quality\ \ factor}$$

Because Q varies along the path of seismic waves, we must estimate that parameter by integration along the path:

$$t^* = \int \frac{dt}{Q} = \sum_{ii=1}^{N} \frac{\Delta t_i}{Q_i}$$



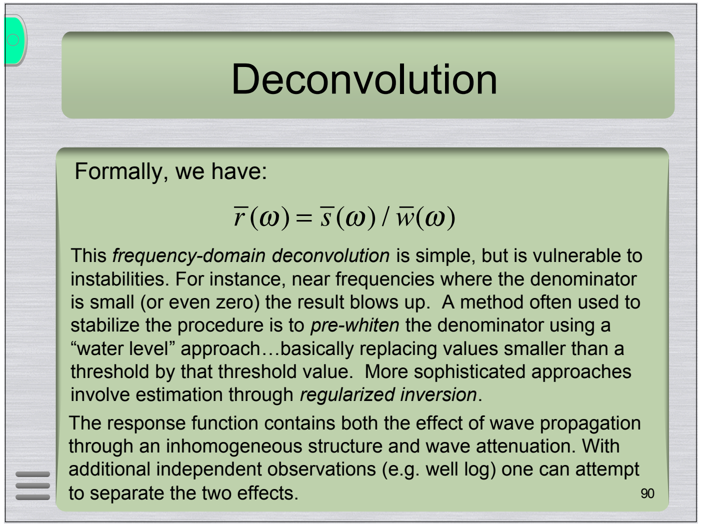
Using linearity

For linear wave propagation (weak motions) attenuation effects can be handled using the usual tools: Green function, transfer function, convolution & deconvolution, etc.

A seismogram s(t) can then be written as the convolution of a source or input pulse w(t) with a response function r(t). For a layered structure, the latter is often formulated as a reflector series.

$$s(t) = w(t) * r(t) \quad ; \quad \overline{s}(\omega) = \overline{w}(\omega)\overline{r}(\omega)$$

If w(t) and s(t) can both be measured independently, the response function is obtained by **deconvolution**



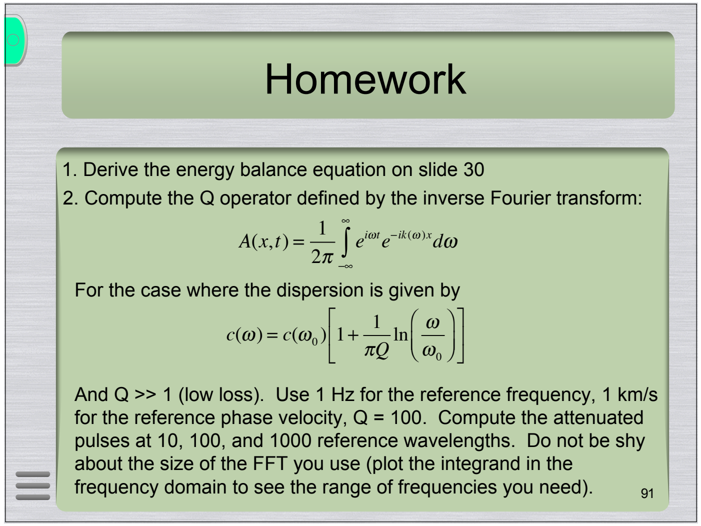
Deconvolution

Formally, we have:

$$\overline{r}(\omega) = \overline{s}(\omega) / \overline{w}(\omega)$$

This *frequency-domain deconvolution* is simple, but is vulnerable to instabilities. For instance, near frequencies where the denominator is small (or even zero) the result blows up. A method often used to stabilize the procedure is to *pre-whiten* the denominator using a "water level" approach…basically replacing values smaller than a threshold by that threshold value. More sophisticated approaches involve estimation through *regularized inversion*.

The response function contains both the effect of wave propagation through an inhomogeneous structure and wave attenuation. With additional independent observations (e.g. well log) one can attempt to separate the two effects.



Homework

1. Derive the energy balance equation on slide 30
2. Compute the Q operator defined by the inverse Fourier transform:

$$A(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} e^{-ik(\omega)x} d\omega$$

For the case where the dispersion is given by

$$c(\omega) = c(\omega_0)\left[1 + \frac{1}{\pi Q}\ln\left(\frac{\omega}{\omega_0}\right)\right]$$

And Q >> 1 (low loss). Use 1 Hz for the reference frequency, 1 km/s for the reference phase velocity, Q = 100. Compute the attenuated pulses at 10, 100, and 1000 reference wavelengths. Do not be shy about the size of the FFT you use (plot the integrand in the frequency domain to see the range of frequencies you need).